Connectivity Graph-Codes

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Abstract

The symmetric difference of two graphs G_1, G_2 on the same set of vertices V is the graph on V whose set of edges are all edges that belong to exactly one of the two graphs G_1, G_2. For a fixed graph H call a collection $\mathcal{G}$ of spanning subgraphs of H a connectivity code for H if the symmetric difference of any two distinct subgraphs in $\mathcal{G}$ is a connected spanning subgraph of H. It is easy to see that the maximum possible cardinality of such a collection is at most $2^{k'(H)} \leq 2^{\delta(H)}$, where $k'(H)$ is the edge-connectivity of H and $\delta(H)$ is its minimum degree. We show that equality holds for any d-regular (mild) expander, and observe that equality does not hold in several natural examples including powers of long cycles and products of a small clique with a long cycle.

1 Introduction

The *symmetric difference* of two graph $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices V is the graph $(V, E_1 \oplus E_2)$ where $E_1 \oplus E_2$ is the symmetric difference between E_1 and E_2, that is, the set of all edges that belong to exactly one of the two graphs.

An intriguing variant of the well studied theory of error correcting codes (see, e.g., [9]) is the investigation of collections $\mathcal{G}$ of graphs on the set of vertices V in which the symmetric difference of every distinct pair satisfies a prescribed property. The systematic study of this topic was initiated in [3], see also [2], [6] for two recent subsequent papers. If all graphs in the collection $\mathcal{G}$ are subgraphs of a fixed graph H, and the property considered is connectivity, we call $\mathcal{G}$ a connectivity code for H. Let $m(H)$ denote the maximum possible cardinality of a connectivity code for H. It is clear that no two distinct members of such a code $\mathcal{G}$ can have exactly the same intersection with the set of edges of any nontrivial cut of H, implying that $m(H) \leq 2^{k'(H)}$, where $k'(H)$ is the edge-connectivity of H. In [3] it is shown that equality holds if H is the complete graph K_n, that is, $m(K_n) = 2^{n-1}$. In [6] it is proved that equality holds also for the 3 by 3 torus $C_3 \times C_3$. This is the (Cartesian) product of two cycles of length 3 in which two vertices are adjacent iff they are equal in

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one coordinate and adjacent in the other. The edge connectivity (and degree of regularity) here is 4, and it is shown in [6] that $m(C_3 \times C_3) = 2^4 = 16$.

Our main result in this note is that for any d-regular graph H satisfying appropriate expansion properties, $m(H) = 2^d$.

Theorem 1.1. *There exists an absolute constant c so that the following holds. Let d and n be integers and let H be a d-regular graph on n vertices. Suppose that for every connected induced subgraph of H on a set W of $w \leq n/2$ vertices, there are at least $cw \log d$ edges connecting W to its complement. Then $m(H) = 2^d$, that is, the maximum cardinality of a connectivity code for H is 2^d.*

We also observe that for several natural examples of d-regular graphs H which are d-edge-connected, $m(H) \leq 2^{d-1}$. In particular, for all $t > 36$, $m(C_3 \times C_t) \leq 8$. This answers a problem suggested in [6].

The proofs are presented in the next sections. Throughout the note, all logarithms are in base 2, unless otherwise specified. To simplify the presentation we omit all floor and ceiling signs, whenever these are not crucial.

2 Expanders

In this section we prove Theorem 1.1. We make no attempt to optimize the absolute constant c in the statement and the related constants in the proofs. The code we construct is a linear code. We start with the following simple lemma.

Lemma 2.1. *Let $H = (V, E)$ be a graph. Assign each edge $e \in E$ a vector $v(e) \in F$, where F is a vector space of dimension r over Z_2. Suppose that for every cut $(S, V - S) = \{e \in E : e \cap S \neq \emptyset, e \cap (V - S) \neq \emptyset\}$ the set of vectors $\{v(e), e \in (S, V - S)\}$ spans F. Then $m(H) \geq 2^r$.*

Proof. Choose a basis of r vectors of F and express each vector $v(e)$ as a linear combination of the elements of this basis. In this expression $v(e)$ is a vector in Z_2^r. For each vector $u \in Z_2^r$, let G_u be the subgraph of H consisting of all edges $e \in E$ for which the inner product of u and $v(e)$ (over Z_2) is nonzero (that is, 1). The symmetric difference of any two distinct graphs G_u and $G_{u'}$ consists of all edges $e \in E$ for which the inner product of $v(e)$ with the nonzero vector $u \oplus u'$ is nonzero. Since for every cut of H the vectors $v(e)$ for e in the cut span F, this symmetric difference must have at least one edge in each cut, implying it is connected. □

Remark: It is not difficult to see that the condition in the last lemma is equivalent to the existence of a *linear* connectivity code of size 2^r for H. This equivalence is not needed for our purpose here.

In order to prove Theorem 1.1 using the above lemma our objective is to show that for any graph H as in the theorem it is possible to assign each edge e a vector $v(e) \in Z_2^d$ so that the vectors assigned to the edges of each cut of H span Z_2^d. In particular, the vectors assigned to all edges incident with any single vertex must form a basis. The expansion properties of the graph ensure that the cuts which are not 1-vertex cuts have significantly more edges than the 1-vertex cuts, and hence it seems intuitively simpler to ensure their vectors span the whole space. The rigorous proof is probabilistic, assigning vectors randomly to most (but not to all) of the edges of H. Since, however, the probability that d random vectors of Z_2^d form a basis is exactly $\prod_{i=1}^{d}(1-2^{-i})$, which is bounded away from 1 (indeed smaller than 1/2), special care is needed to ensure that the vectors assigned to all edges in every 1-vertex cut form a basis. To do so, we do not assign vectors randomly to all edges, but only to most of them, and complete the assignment using the following lemma.

Lemma 2.2. *Let $H = (V, E)$ be a d-regular graph, let E' be a subset of E and let $v(e), e \in E'$ be an assignment of a vector in Z_2^d for any edge $e \in E'$. Suppose that for every vertex u the set of vectors $v(e)$ assigned to all edges in E' that are incident with u is linearly independent. Then it is possible to complete the given partial assignment by assigning a vector $v(e) \in Z_2^r$ to every edge $e \in E - E'$, so that for every vertex u, the set of vectors $v(e)$ assigned to all d edges incident with it forms a basis of Z_2^d.*

Proof. Define the required vectors greedily in an arbitrary order, maintaining the property that the vectors assigned to all edges incident with any vertex are linearly independent. When we have to assign a vector to an edge uu' there are at most $2^{d-1}-1$ nonzero vectors that are forbidden in order to ensure that the vectors assigned to edges incident with u will stay independent. Similarly, u' forbids at most $2^{d-1}-1$ nonzero vectors. Since $2(2^{d-1}-1) < 2^d - 1$ there is always a way to choose a vector that can be assigned to uu' maintaining the required property. This completes the proof. □

We also need the following immediate consequence of Petersen's Theorem.

Lemma 2.3. *For every even k and every $d \geq k$, any d-regular graph contains a spanning subgraph in which every degree is either k or $k-1$*

Proof. If d is even this follows by a repeated application of Petersen's Theorem [12] that asserts that any d-regular (multi)graph contains a 2-factor. If d is odd, add to it a perfect matching (repeating existing edges if needed), apply the previous case to the resulting graph, and remove the edges of the added matching chosen to the spanning subgraph. □

The main technical lemma we need for the proof of Theorem 1.1 will be established using the (asymmetric) Lovász Local Lemma, which we state next.

Lemma 2.4 (The Lovász Local Lemma, c.f., [4], Chapter 5)**.** *Let $A_i, i \in I$ be a finite collection of events in an arbitrary probability space. A dependency graph for these events is a graph D whose set of vertices is the events A_i, where each event is mutually independent of all the events that are its non-neighbors. Let D be such a dependency graph, and let $N(A_i)$ denote the set of all neighbors of A_i in D. Suppose that for each event A_i there is a real number $x_i \in [0,1)$ so that for each $i \in I$*

$$Prob(A_i) \leq x_i \prod_{j \in I, A_j \in N(A_i)} (1 - x_j)$$

Then, with positive probability none of the events A_i occurs.

We can now state and prove the main lemma, in which the constants 1000 and 8 can be easily improved.

Lemma 2.5. *Let $H = (V, E)$ be a d-regular graph on a set V of n vertices, where $d \geq 1000$. Suppose that for every connected induced subgraph of H on a set W of $w \leq n/2$ vertices, there are at least $w(8 \log d + 2)$ edges connecting W to its complement. Then there is a spanning subgraph $H' = (V, E')$ of H in which the degree of every vertex is at least $d - 3\log d - 2$ and an assignment of a vector $v(e) \in Z_2^d$ for every edge $e \in E'$ so that the following two properties hold.*

1. *For every vertex $u \in V$, the set of vectors $v(e)$ assigned to the edges of H' incident with u is linearly independent.*
2. *For every integer w, $2 \leq w \leq n/2$, and for every set W of w vertices so that the induced subgraph of H on W is connected (in H), the set of vectors $v(e)$ for the vectors e in H' that connect W to its complement span Z_2^d.*

Proof. Let k be the smallest even number which is at least $d - 3\log d - 1$. By Lemma 2.3 there is a spanning subgraph $H' = (V, E')$ of H in which every degree is either k or $k-1$. Thus every degree of H' is at least $d - 3\log d - 2$ and at most $d - 3\log d$.

Fixing this subgraph H', choose for every edge e of H', independently, a vector $v(e) \in Z_2^d$ uniformly at random among all 2^d vectors of Z_2^d (including the 0 vector, to simplify the computation). To complete the proof we show, using the asymmetric Lovász Local Lemma (Lemma 2.4 above) that with positive probability this random choice satisfies the properties stated in the lemma. We proceed with the details.

For each vertex $u \in V$, let $A(u)$ denote the event that the set of random vectors $v(e)$ assigned to the edges of H' incident with u are not linearly independent.

For every integer w, $2 \leq w \leq n/2$, and for every set W of w vertices of H such that the induced subgraph of H on W is connected (in H), let $B(W)$ denote the event that the set of vectors $v(e)$ for the edges e in H' that connect W to its complement does not span Z_2^d. If $|W| = w$ call $B(W)$ an event of type w.

It is not difficult to upper bound the probabilities of these events. For each vertex $u \in V$,

$$\text{Prob}(A(u)) \leq 2^{-3\log d}. \tag{1}$$

Indeed, let the edges of H' incident with u be $e_1, e_2, \ldots, e_s$, where the numbering is arbitrary. Then $s \leq d - 3\log d$. The probability that the vector $v(e_i)$ lies in the span of the previous vectors $v(e_1), \ldots, v(e_{i-1})$ is at most $2^{i-1}/2^d$. The required estimate follows by summing over all i, using the fact that $s \leq d - 3\log d$.

Next we show that for every event $B(W)$ of type w

$$\text{Prob}(B(W)) \leq 2^{-4w\log d}. \tag{2}$$

It is convenient to split the possible values of w into two ranges. If $2 \leq w \leq d/5$ then every vertex of W has at least $d - w + 1 > 0.8d$ neighbors in H that do not lie in W. Among these edges, at most $3\log d + 2$ edges incident with each vertex of W belong to H and not to H', implying that there are at least $w(0.8d - 3\log d - 2)$ edges of H' connecting vertices of W to its complement. For every nonzero vector $z \in Z_2^d$, the probability that all vectors $v(e)$ corresponding to these edges are orthogonal to z is at most $2^{-w(0.8d-3\log d-2)} \leq 2^{-(d+4w\log d)}$, where here we used the fact that for $d \geq 1000$ and $w \geq 2$,

$$w(0.8d - 3\log d - 2) \geq d + 4w\log d.$$

The desired estimate for this case follows by the union bound over all $2^d - 1 < 2^d$ choices for the vector z.

If $d/5 \leq w \leq n/2$ then, by assumption, there are at least $w(8\log d + 2)$ edges of H connecting W and its complement. Among these edges, at most $3\log d + 2$ edges incident with each vertex of W belong to H and not to H', implying that there are at least $5w\log d$ edges of H' that connect W and its complement. For every nonzero vector $z \in Z_2^d$, the probability that all vectors $v(e)$ corresponding to these edges are orthogonal to z is at most $2^{-5w\log d} \leq 2^{-(d+4w\log d)}$ where the last inequality holds since for $w \geq d/5$ and $d \geq 1000$, $5w\log d \geq d + 4w\log d$. The desired estimate follows, as in the previous case, by the union bound over all $2^d - 1 < 2^d$ choices for the vector z.

In order to apply the local lemma we need to define a dependency graph D for all events $A(u), B(W)$. To do so we need the known fact (c.f., e.g., [1]) that for every graph with maximum degree d, for every integer w, and for every vertex u of the graph, the number of sets of w vertices that contain u and induce a connected subgraph is smaller than $(ed)^w$. Note that each event $A(u)$ is determined by the random vectors assigned to the edges of H' incident with it. Similarly, each event $B(W)$ is determined by the random vectors assigned to the edges of H' connecting W and its complement. It is clear that the graph on the events in which two events are connected iff the edge sets whose vectors determine them intersect is a dependency graph.

It follows that each event $A(u)$ is independent of all other events besides at most d other events $A(u')$ and besides at most $d(ed)^w < d^{2w}$ events $B(W)$ of type w, for each $2 \leq w \leq n/2$. Similarly, each event $B(W)$ of type w is independent of all other events besides at most wd events of the form $A(u)$ and at most $wd(ed)^r < wd^{2r}$ events $B(W')$ corresponding to sets W of size r, for every $2 \leq r \leq n/2$. This is because the set of all edges of H' connecting a vertex in W with one in its complement cover at most w vertices of W and less than $(d-1)w$ vertices of its complement, and each such vertex lies in at most $(ed)^r$ subsets of size r that induce a connected subgraph of H.

To apply the local lemma define, for each event $A(u)$, a real $x_u = 2^{-2\log d} = \frac{1}{d^2}$. For each event $B(W)$ of type w, define $x_W = 2^{-3w\log d} = \frac{1}{d^{3w}}$. To complete the proof using Lemma 2.4 it remains to check the following inequalities.

1. For every vertex u,

$$\text{Prob}(A(u)) \leq \frac{1}{d^2}(1-\frac{1}{d^2})^d \prod_{w\geq 2}(1-\frac{1}{d^{3w}})^{d^{2w}} \tag{3}$$

2. For every event $B(W)$ of type w

$$\text{Prob}(B(W)) \leq \frac{1}{d^{3w}}(1-\frac{1}{d^2})^{dw} \prod_{r\geq 2}(1-\frac{1}{d^{3r}})^{wd^{2r}}. \tag{4}$$

Inequality (3) follows from (1) and the fact that

$$(1-\frac{1}{d^2})^d \prod_{w\geq 2}(1-\frac{1}{d^{3w}})^{d^{2w}} \geq (1-\frac{1}{d})\prod_{w\geq 2}(1-\frac{1}{d^w}) \geq 1-\sum_{w\geq 1}\frac{1}{d^w} > 1/2 > 1/d.$$

Inequality (4) follows from (2) and the fact that

$$(1-\frac{1}{d^2})^{dw}\prod_{r\geq 2}(1-\frac{1}{d^{3r}})^{wd^{2r}} \geq e^{-2w/d}\prod_{r\geq 2}e^{-2w/d^r} = e^{-2w\sum_{r\geq 1}1/d^r} \geq e^{-w} > d^{-w}.$$

This completes the proof of the lemma. □

The proof of the main result, Theorem 1.1, follows quickly from the assertion of the previous lemmas, as shown next.

Proof. Let $H=(V,E)$ be a graph satisfying the assumptions of Theorem 1.1 with, say, $c=100$. Note that by the assumption $100\log d \leq d$ implying that $d \geq 1000$. (It is worth noting also that the proof works, as it is, for $c=9$ and the additional assumption that $d \geq 1000$.) By Lemma 2.1 it suffices to show that there is an assignment of a vector $v(e) \in Z_2^d$ for every edge $e \in E$, so that the vectors assigned to the edges of any cut $(S, V-S)$ of H span Z_2^d. Since any cut contains all edges of a cut in which at least one side induces a connected subgraph of at most half the vertices, it suffices to ensure that for every such cut the vectors assigned to its edges span Z_2^d. By Lemma 2.5 there is a spanning subgraph H' of H and an assignment of a vector in Z_2^d to each of its edges so that the conclusion of Lemma 2.5 hold. Lemma 2.2 ensures that this partial assignment of vectors can be completed to an assignment of vectors $v(e) \in Z_2^d$ for every edge e of H so that the vectors assigned to the edges of any 1-vertex cut form a basis. The vectors assigned to the edges of any other cut still span, of course, Z_2^d, as they contain all vectors of edges of H' that belong to the cut, which span Z_2^d. This completes the proof. □

3 Graphs admitting only smaller codes

In this section we observe that there are natural classes of d-regular graphs H with edge connectivity d so that $m(H)$ is strictly smaller than 2^d (and is in fact at most half of that). This follows from the following simple lemma.

Lemma 3.1. *Let $H = (V, E)$ be a graph, and let $E_1, E_2, \ldots, E_s$ be sets of edges of H, where each E_i contains at most t edges. Suppose that for every $1 \leq i < j \leq s$, the set of edges $E_i \cup E_j$ disconnects H, that is, $H - (E_i \cup E_j)$ is disconnected. If $s > (2^t + 1)2^{t-1}$ then $m(H) \leq 2^t$.*

Proof. Let $\mathcal{G}$ be a family of $2^t + 1$ spanning subgraphs of H. We have to show that it must contain two distinct members whose symmetric difference is disconnected. By the pigeonhole principle, for each fixed i, $1 \leq i \leq s$, there is a pair $\{G_1^{(i)}, G_2^{(i)}\}$ of distinct members of $\mathcal{G}$ that have the same intersection with the set E_i. By a second application of the pigeonhole principle, since $s > \binom{|\mathcal{G}|}{2}$, there are distinct i, j so that

$$\{G_1^{(i)}, G_2^{(i)}\} = \{G_1^{(j)}, G_2^{(j)}\}.$$

Therefore, the symmetric difference of these two graphs does not contain any edge of $E_i \cup E_j$ and is thus disconnected. □

We next describe two families of regular graphs for which the above lemma implies that the maximum size of a connectivity code is strictly smaller than the trivial bound that follows from the edge-connectivity.

The (Cartesian, or box) product $H_1 \times H_2$ of two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ is the graph whose vertex set is the set $V_1 \times V_2$ where (v_1, v_2) and (u_1, u_2) are connected iff either $v_1 = u_1$ and v_2, u_2 are connected in H_2 or $v_2 = u_2$ and v_1, u_1 are connected in H_1.

Proposition 3.2. *For every clique K_t and cycle C_s, where $s > (2^t + 1)2^{t-1}$, the graph $H_{t,s}$ is $(t + 1)$-regular and its edge connectivity is $t + 1$, but $m(H_{t,s}) \leq 2^t$.*

Proof. It is clear that $H_{t,s}$ is $(t+1)$-regular. The fact that its edge connectivity is also $t+1$ is a consequence of the well known fact that the edge connectivity of a product is at least the sum of the edge connectivities of the two factors (see, e.g., [8]). The upper bound for $m(H_{t,s})$ follows from Lemma 3.1 in which the s sets E_i are the sets of edges connecting the vertices of $H_{t,s}$ in which the second coordinate is vertex number i of the cycle C_s and the vertices in which the second coordinate is vertex number $i+1$ of the cycle (where addition is modulo s). □

Since $K_3 = C_3$, a special case of the above result is that for all $s > 36$, the maximum possible cardinality of a connectivity code for the torus $C_3 \times C_s$ satisfies $m(C_3 \times C_s) \leq 8$. This answers a question raised in [6], (although the problem of determining $m(C_t \times C_s)$ for all t, s remains open.)

Proposition 3.3. *For $s > 2k$ let $C_s^{(k)}$ denote the graph obtained from the cycle C_s by connecting any two vertices of distance at most k in the cycle. Then $C_s^{(k)}$ is $2k$-regular, its edge-connectivity is $2k$, and if $s > (2^k+1)2^{k-1}$ then $m(C_s^{(k)}) \leq 2^{2k-1}$*

Proof. It is clear that $C_s^{(k)}$ is $2k$-regular. The fact that it is $2k$-edge connected (and in fact even $2k$ (vertex) connected) is known, c.f., e.g., [7] pp 48-49. In order to prove an upper bound for $m(C_s^{(k)})$ we apply Lemma 3.1. For each edge f of the cycle C_s, let E_f denote the set of all edges $\{u, v\}$ of the graph $C_s^{(k)}$ for which the unique shortest path in C_s connecting u and v includes the edge f. Then $|E_f| = 2k-1$ for each of the s edges f of C_s. It is easy to check that for any two distinct f, f', $E_f \cup E_{f'}$ disconnects $C_s^{(k)}$. The required upper bound for $m(C_s^{(k)})$ thus follows from Lemma 3.1. □

4 Concluding remarks and open problems

- An (n, d, λ)-graph H is a d-regular graph on n vertices in which the absolute value of each nontrivial eigenvalue is at most λ. It is well known that if λ is significantly smaller than d then any such H has strong expansion properties. In fact, it suffices to assume that the second largest eigenvalue of H is at most λ (with no assumption about the most negative eigenvalue). A simple result proved in [5] is that for any set W of $w \leq n/2$ vertices of a d-regular graph on n vertices in which the second largest eigenvalue is at most λ there are at least $w(d-\lambda)/2$ edges connecting W and its complement. Theorem 1.1 thus implies that if $\lambda \leq d - 2c \log d$ then $m(H) = 2^d$. Note that this is a pretty mild assumption on λ, as it is known that for every d there are infinitely many bipartite d-regular graphs in which the second largest eigenvalue is at most $2\sqrt{d-1}$, see [10].

- Our proof of Theorem 1.1 provides a linear connectivity code of maximum possible cardinality for any graph H satisfying the assumptions. It will be interesting to decide if there are interesting examples of graphs H for which non-linear connectivity codes can be larger than linear ones. We note that the code of maximum possible cardinality for the complete graph K_n described in [3] is linear, and so is the code of maximum possible cardinality for $C_3 \times C_3$ given in [6].

- It may be interesting to study the computational problem of computing or estimating $m(H)$ for a given input graph H. As mentioned in the introduction, $m(H)$ is always at most $2^{k'(H)}$, where $k'(H)$ is the edge connectivity of H. On the other hand, $m(H)$ is always at least $2^{\lfloor k'(H)/2 \rfloor}$. This is because an immediate consequence of the known result of Nash-Williams about packing edge-disjoint tress in graphs [11] implies that H has at least $k = \lfloor k'(H)/2 \rfloor$ pairwise edge disjoint spanning trees T_i. The collection of all 2^k unions $\cup_{i \in I} E(T_i)$ of the edge sets of any subset I of these trees is a (linear) connectivity code for H. Since $k'(H)$ can be computed in polynomial time, this supplies an efficient algorithm for approximating the logarithm of $m(H)$ up to a factor of (roughly) 2.

- By the remark in the previous comment, the smallest possible value of $m(H)$ for a graph H with an even edge connectivity $k' = 2k$ is at least 2^k. It is not too difficult to give examples showing that this is tight. Indeed, let s be an integer, $s > 2^{k-1}(2^k + 1)$, and let $H = H(s, k)$ be a graph obtained from the vertex disjoint union of s cliques $K(0), K(1), \ldots K(s-1)$, each of size $2k+1$, by adding a matching M_i of k edges between $K(i)$ and $K(i+1)$, for all $0 \leq i \leq s-1$, where $K(s) = K(0)$. It is easy to see that the edge connectivity of this graph is $2k$. Indeed, deleting less than $2k$ edges leaves each of the cliques $K(i)$ connected and leaves at least one edge of every matching M_i besides at most one, keeping the graph connected. Thus the edge connectivity is at least $2k$ and thus $m(H) \geq 2^k$. On the other hand, any union of two of the matchings M_i disconnects H and hence the edge connectivity is exactly $2k$. In addition, by Lemma 3.1, $m(H) \leq 2^k$ and therefore $m(H) = 2^k$.

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