

# An Elementary Construction of Constant-Degree Expanders

Noga Alon \*

Oded Schwartz †

Asaf Shapira ‡

## Abstract

We describe a short and easy to analyze construction of constant-degree expanders. The construction relies on the replacement product, applied by [14] to give an iterative construction of bounded-degree expanders. Here we give a simpler construction, which applies the replacement product (only twice!) to turn the Cayley expanders of [4], whose degree is  $\text{polylog } n$ , into constant degree expanders. This enables us to prove the required expansion using a new simple combinatorial analysis of the replacement product (instead of the spectral analysis used in [14]).

## 1 Introduction

All graphs considered here are finite, undirected and may contain self-loops and parallel edges. Expanders are graphs, which are simultaneously sparse, yet highly connected, in the sense that every cut contains (relatively) many edges. In this note we mostly work with the notion of *edge-expansion*. A  $d$ -regular graph  $G = (V, E)$  is a  $\delta$ -edge-expander ( $\delta$ -expander for short) if for every set  $S \subseteq V$  of size at most  $\frac{1}{2}|V|$  there are at least  $\delta d|S|$  edges connecting  $S$  and  $\bar{S} = V \setminus S$ , that is,  $e(S, \bar{S}) \geq \delta d|S|$ . For brevity, we say that a graph is an  $[n, d, \delta]$ -expander if it is an  $n$ -vertex  $d$ -regular  $\delta$ -expander. Expanders are some of the most widely used objects in theoretical computer science, and have also found many applications in other areas of computer-science and mathematics. See the survey of Hoory et. al. [9] for a discussion of several applications and references. Another widely used notion of expansion is based on algebraic properties of a matrix representation of the graph. Let  $G = (V, E)$  be an  $n$ -vertex  $d$ -regular graph, and let  $A$  be the adjacency matrix of  $G$ , that is, the  $n \times n$  matrix, with  $A_{i,j}$  being the number of edges between  $i$  and  $j$ . It is easy to see that  $\mathbf{1}^n$  is an eigenvector of  $A$

with eigenvalue  $d$ , and that this is the only eigenvector with this eigenvalue iff  $G$  is connected. We denote by  $\lambda_2(G)$  the second largest eigenvalue of  $A$ . It is easy to see that  $\lambda_2(G) = \max_{0 \neq x \perp \mathbf{1}^n} \langle Ax, x \rangle / \langle x, x \rangle$ . The following is a well known relation between the expansion of  $G$  and  $\lambda_2(G)$ .

**THEOREM 1.1.** ([2], [3], [6]) *Let  $G$  be a  $\delta$ -expander with adjacency matrix  $A$  and let  $\lambda_2 = \lambda_2(G)$  be the second largest eigenvalue of  $A$ . Then,  $\frac{1}{2}(1 - \lambda_2/d) \leq \delta \leq \sqrt{2(1 - \lambda_2/d)}$ .*

Our construction uses only the first simple inequality, but for completeness, we include a very short proof of the second direction of this theorem in the appendix.

The most useful expanders are those with constant degree. A priori, it is not clear that constant-degree expanders even exist. Pinsker [13] established their existence.

**THEOREM 1.2.** ([13]) *There exists a fixed  $\delta > 0$ , such that for any  $d \geq 3$  and even integer  $n$ , there is an  $[n, d, \delta]$ -expander, which is  $d$ -edge-colorable <sup>1</sup>.*

One way to prove the above is to take a random  $d$ -regular bipartite graph. In most applications however one needs to efficiently construct constant degree expanders. There are two notions of constructibility of  $d$ -regular expanders. The first (weaker) notion requires the  $n$ -vertex graph to be constructible in polynomial time in its size. The second (stronger) notion requires that given a vertex  $v$  and  $i \in [d]$  it would be possible to generate the  $i^{\text{th}}$  neighbor of  $v$  in time  $\text{poly}(\log n)$ . Such an expander is said to be *fully explicit*. In applications, where one needs to use the entire graph, it is often enough to use the weaker notion. However, in such cases (e.g. in certain reductions) one frequently needs to be able to construct a graph of a given size  $n$ . It has been observed by many that to this end it is enough to be able to construct graphs of size  $\Theta(n)$  (e.g., one can take a  $cn$ -vertex expander and join groups of  $c$  vertices to get an  $n$ -vertex expander with positive expansion). In other cases, where one needs only part of the expander (e.g.,

\*Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. Email: nogaa@tau.ac.il. Research supported in part by a USA Israeli BSF grant, by a grant from the Israel Science Foundation, and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University.

†School of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: odedsc@tau.ac.il.

‡Microsoft Research. Email: asafico@tau.ac.il.

<sup>1</sup>That is, one can assign its edges  $d$  colors such that edges incident with the same vertex are assigned distinct colors.

when performing a random walk on a large expander) one usually needs the stronger notion of fully explicitness. However, in these cases it is usually enough to be able to construct an expander of size  $\text{poly}(n)$ , as what we are interested in is actually the logarithm of the size of the graph. Margulis [11] and Gabber and Galil [7] were the first to efficiently construct constant degree expanders. Following was a sequence of works that culminated in the construction of Lubotzky, Phillips and Sarnak [10] and Margulis [12] of Ramanujan Graphs. These constructions rely (directly or indirectly) on estimations of the second largest eigenvalue of the graphs, and some of them, rely on deep mathematical results. A simpler, iterative construction was given by Reingold, Vadhan and Wigderson [14]. This construction also relies on proving the expansion of the graphs by estimating their eigenvalues, and is the first construction of constant degree expanders with relatively elementary analysis. Additional algorithms for constructing bounded-degree expanders appear in [1] and [5], but these should be viewed as algorithms rather than explicit constructions, and in particular are not fully explicit in the sense described above.

Our construction is based on the replacement product of two graphs  $G$  and  $H$ , which is one of the most natural ways of combining two graphs. We start by defining this basic operation.

**DEFINITION 1.1.** *Let  $G$  be a  $D$ -regular  $D$ -edge-colorable graph on  $n$  vertices and let  $H$  be a  $d$ -regular graph on  $D$  vertices. Suppose  $G$  is already equipped with a proper  $D$ -edge-coloring. The replacement product  $G \circ H$  is the following  $2d$ -regular graph on  $nD$  vertices: We first replace every vertex  $v_i$  of  $G$  with a cluster of  $D$  vertices, which we denote  $C_i = \{v_1^i, \dots, v_D^i\}$ . For every  $1 \leq i \leq n$  we put a copy of  $H$  on  $C_i$  by connecting  $v_p^i$  to  $v_q^i$  if and only if  $(p, q) \in E(H)$ . Finally, for every  $(p, q) \in E(G)$ , which is colored  $t$ , we put  $d$  parallel edges between  $v_p^t$  and  $v_q^t$ .*

Note that if  $H$  is  $d$ -edge-colorable then  $G \circ H$  is  $2d$ -edge colorable: simply color the copies of  $H$  within each set  $C_i$  using colors  $1, \dots, d$ . As the edges between the sets  $C_i$  form  $d$  parallel copies of a perfect matching on the vertices of  $G \circ H$ , we can color any set of  $d$  parallel edges using the colors  $d+1, \dots, 2d$ . Already in the 80's, Gromov [8] has analyzed the effect of (a slight variant of) this operation on the spectral properties of graphs. Reingold, Vadhan and Wigderson [14] considered the above variant, and showed, via a reduction to their algebraic analysis of the zig-zag product, that if two graphs are expanders then so is their product. Their argument is based on analyzing  $\lambda(G \circ H)$  as a function of  $\lambda(G)$  and  $\lambda(H)$ . We analyze the replacement product

directly via an elementary combinatorial argument.

**THEOREM 1.3.** *Suppose  $E_1$  is an  $[n, D, \delta_1]$ -expander and  $E_2$  is a  $[D, d, \delta_2]$ -expander. Then,  $E_1 \circ E_2$  is an  $[nD, 2d, \frac{1}{80}\delta_1^2\delta_2]$ -expander.*

The proof of Theorem 1.3 is completely simple; we show that  $e(X, \bar{X})$  has either many edges within the clusters  $C_i$  or between them. Our main result is a new construction of constant-degree expanders. The main idea can be summarized as follows: a simple special case of one of the results of [4] gives a construction of  $[n, O(\log^2 n), \frac{1}{4}]$ -expanders. To get expanders with constant degree we construct such an  $[n, O(\log^2 n), \frac{1}{4}]$ -expander and then apply the replacement product with another similar expander in order to reduce the degree to  $O(\sqrt{\log n})$ . We now find a constant degree expander of size  $O(\sqrt{\log n})$ , using exhaustive search, and apply a final replacement product to get a constant degree. Note that here we do not care much about the fact that the replacement product decreases the edge-expansion as we only apply it twice. A suitable choice of parameters gives the following construction, whose analysis relies solely on the easy part of Theorem 1.1, a simple special case of the result of [4] and on the elementary analysis of the replacement product given in Theorem 1.3.

**THEOREM 1.4. (MAIN RESULT)** *There exists a fixed  $\delta > 0$  such that for any integer  $q = 2^t$  and for any  $q^4/40 \leq r \leq q^4/2$  there is a polynomial time constructible  $[q^{4r+12}, 12, \delta]$ -expander.*

For completeness we prove all the necessary ingredients, thus obtaining a short and self-contained construction of constant-degree expanders. It is easy to see that given  $n$ , Theorem 1.4 can be used to construct an  $m$ -vertex expander with  $n \leq m = O(n \log n)$ . The construction and its analysis appear in the following section. In Section 3 we observe that simple variants of Theorem 1.4 give a construction with  $\Theta(n)$  vertices and a construction which is fully explicit.

## 2 The Construction

Let us start by describing the special case of [4] that suffices for our purposes. For any  $q = 2^t$  and  $r \in \mathbb{N}$ , we define a graph  $LD(q, r)$  as follows. The vertices are all elements of  $\mathbb{F}_q^{r+1}$ , which can be thought of as all strings  $(a_0, \dots, a_r) \in \mathbb{F}_q^{r+1}$ . A neighbor of a vertex  $a$  is indexed by an element  $(x, y) \in \mathbb{F}_q^2$ . In this notation neighbor  $(x, y)$  of vertex  $a = (a_0, \dots, a_r)$  is  $a + y \cdot (1, x, x^2, \dots, x^r)$ .  $LD(q, r)$  is clearly a  $q^2$ -regular graph on  $q^{r+1}$  vertices. It is also  $q^2$ -edge-colorable as we can color the edges indexed  $(x, y)$  using the "color"  $(x, y)$  (note that this is well defined as addition and subtraction are identical in

$\mathbb{F}_{2^t}$ ). The following result is a special case of the result of [4]:

**THEOREM 2.1.** ([4]) *For any  $q = 2^t$  and integer  $r < q$  we have  $\lambda_2(LD(q, r)) \leq rq$ .*

Note that the above theorem, together with the left inequality of Theorem 1.1, imply that if  $r \leq q/2$  then  $LD(q, r)$  is a  $[q^{r+1}, q^2, \frac{1}{4}]$ -expander. We first prove our main result based on Theorems 1.3, 2.1 and the left inequality of Theorem 1.1. We then prove these three results.

**Proof of Theorem 1.4:** Given integers  $q$  and  $q^4/40 \leq r \leq q^4/2$ , we start by enumerating all 3-regular graphs on  $q^2$  vertices until we find one which is a  $\delta$ -expander and 3-edge colorable (one exists by Theorem 1.2). This step can clearly be carried out in time  $q^{O(q^2)}$ . Denote by  $E_1$  the expander we find and define  $E_3 = LD(q^4, r)$ ,  $E_2 = LD(q, 5)$  and set  $E_4 = E_3 \circ (E_2 \circ E_1)$  to be our final graph. As  $E_1, E_2$  and  $E_3$  are  $[q^2, 3, \delta]$ ,  $[q^6, q^2, \frac{1}{4}]$  and  $[q^{4r+4}, q^8, \frac{1}{4}]$  expanders respectively,  $E_4$  is a  $[q^{4r+12}, 12, \delta']$ -expander for some absolute constant  $\delta'$  (here we rely on Theorem 1.3). Moreover, given  $E_1$  one can easily compute  $E_4$  in time polynomial<sup>2</sup> in the size of  $E_4$ . As  $r \geq q^4/40$ ,  $E_4$  is of size at least  $q^{q^4/10}$ , thus the first step of finding  $E_1$  also takes time polynomial in the size of  $E_4$ , as needed.

Let us conclude by showing that for any  $n$  we can construct an expander on  $m$  vertices, where  $n \leq m = O(n \log n)$ . As  $r \geq q^4/40$  we have  $q = O(\sqrt[4]{\log n})$ , hence it is enough to show that for any large enough  $n$ , there exists  $q = 2^t$  and  $q^4/40 \leq r_0 \leq q^4/2$  such that  $n/q^4 \leq q^{4r_0+12} \leq n$ . Given  $n$  let  $x$  be such that  $x^{4x^4/40+12} = n$  and let  $x/2 \leq q \leq x$ . By our choice of  $x$  and  $q$  we get the following: if  $r = q^4/40$  we have  $q^{4q^4/40+12} \leq n$ , and if  $r = q^4/2$  then  $q^{4q^4/2+12} \geq (x/2)^{2(x/2)^4+12} \geq x^{4x^4/40+12} = n$ . Therefore, for some  $q^4/40 \leq r_0 \leq q^4/2$  we have  $n/q^4 \leq q^{4r_0+12} \leq n$ . ■

**Proof of Theorem 1.3:** Put  $E_3 = E_1 \circ E_2$  and consider any set  $X$  of vertices in  $E_3$  of size at most  $\frac{1}{2}nD$ . Note that we can view the vertex set of  $E_3$  as composed of  $n$  clusters of vertices  $C_1, \dots, C_n$ , each of size  $D$ . Our goal is to show that there are at least  $\frac{1}{80}\delta_1^2\delta_2 \cdot 2d|X|$  edges leaving  $X$ . We simply show that there are either many edges leaving  $X$  within the sets  $C_i$  or between these sets. Set  $X_i = X \cap C_i$ , let  $I' \subseteq [n]$  be the set of indices of the sets  $X_i$ , whose size is at most  $(1 - \frac{1}{4}\delta_1)D$  and let  $I'' = \{1, \dots, n\} \setminus I'$ . We first consider the contribution

<sup>2</sup>Note that when constructing  $E_2$  and  $E_3$  we need representations of  $\mathbb{F}_q$  and  $\mathbb{F}_{q^4}$ . These representations can be found using exhaustive search in time  $poly(q^4)$  that is much smaller than the size of  $E_4$  and thus negligible.

of the sets  $X_i$  with  $i \in I'$ . As  $E_2$  is a  $\delta_2$ -expander, there are at least  $\frac{1}{4}\delta_1\delta_2d|X_i|$  edges connecting  $X_i$  and  $C_i \setminus X_i$ . Partition  $X$  into two sets  $X'$  and  $X''$  according to  $I'$  and  $I''$  as follows:  $X' = \bigcup_{i \in I'} X_i$  and  $X'' = \bigcup_{i \in I''} X_i$ . By the above, the number of edges connecting  $X'$  and  $\bar{X}$  is at least  $\frac{1}{4}\delta_1\delta_2d|X'|$ . If  $|X'| \geq \frac{1}{10}\delta_1|X|$  then we are done, as this means that there are at least  $\frac{1}{80}\delta_1^2\delta_2 \cdot 2d|X|$  edges connecting  $X$  and its complement  $\bar{X}$ .

Suppose then that  $|X'| \leq \frac{1}{10}\delta_1|X|$ , implying that  $|X''| \geq (1 - \frac{1}{10}\delta_1)|X|$ . We now consider the contribution of the edges leaving the sets  $C_i$ . As the sets  $X_i$  with  $i \in I''$  have size at least  $(1 - \frac{1}{4}\delta_1)D$  we infer that  $|X''|/D \leq |I''| \leq |X''|/(1 - \frac{1}{4}\delta_1)D$ . In particular, as  $|X''| \leq |X| \leq \frac{1}{2}nD$  we have  $|I''| \leq \frac{2}{3}n$ . Therefore, as  $E_1$  is an  $[n, D, \delta_1]$ -expander, there is a set of edges  $M$ , where  $|M| \geq \frac{1}{2}\delta_1D|I''|$ , connecting the vertices of  $I''$  with the vertices of  $I'$ . Let us now consider the corresponding  $d|M| \geq \frac{1}{2}\delta_1dD|I''|$  edges in the graph  $E_3$ . These edges connect vertices from  $\bigcup_{i \in I'} C_i$  with vertices from  $\bigcup_{i \in I''} C_i$ . As each  $X_i$  with  $i \in I''$  is of size at least  $(1 - \frac{1}{4}\delta_1)D$ , we infer that at most  $\frac{1}{4}\delta_1dD|I''|$  of these  $d|M|$  edges connect a vertex in  $C_i \setminus X_i$  with a vertex of  $\bigcup_{i \in I'} C_i$ . Therefore, there are at least  $\frac{1}{4}\delta_1dD|I''|$  edges connecting  $\bigcup_{i \in I''} X_i$  with the vertices of  $\bigcup_{i \in I'} C_i$ . The number of these  $d|M|$  edges that connect vertices from  $\bigcup_{i \in I''} C_i$  with vertices of  $X'$  is clearly at most  $d|X'|$ . As we have  $|X'| \leq \frac{1}{10}\delta_1|X| \leq \frac{1}{6}\delta_1D|I''|$  we infer that there are at most  $\frac{1}{6}\delta_1dD|I''|$  such edges. We conclude that at least  $\frac{1}{12}\delta_1dD|I''|$  edges connect vertices of  $\bigcup_{i \in I''} X_i$  (that belong to  $X$ ) with vertices of  $\bigcup_{i \in I'} C_i \setminus X_i$  (that belong to  $\bar{X}$ ). As  $|I''| \geq |X''|/D$  and  $|X''| \geq \frac{1}{2}|X|$  this means that there are at least  $\frac{1}{48}\delta_12d|X|$  edges connecting  $X$  and  $\bar{X}$ , as needed. ■

**Proof of Theorem 2.1:** Set  $\mathbb{F} = \mathbb{F}_{2^t}$ ,  $n = 2^{t(r+1)}$  and let  $M$  be the  $n \times n$  adjacency matrix of  $LD(2^t, r)$ . Let  $L : \mathbb{F} \rightarrow \{0, 1\}$  be any surjective linear map<sup>3</sup>. Let us describe the eigenvectors of  $M$  over  $\mathbb{R}$ . We will use elements of  $\mathbb{F}^{r+1}$  in order to “name” these vectors as well as to “name” entries of these vectors. For every sequence  $a = (a_0, \dots, a_r) \in \mathbb{F}^{r+1}$ , let  $v_a$  be the vector, whose  $b^{th}$  entry (where  $b \in \mathbb{F}^{r+1}$ ) satisfies  $v_a(b) = (-1)^{L(\sum_{i=0}^r a_i b_i)}$ . It is easy to see that the vectors  $\{v_a\}_{a \in \mathbb{F}^{r+1}}$  are orthogonal, therefore these are the only eigenvectors of  $M$ . Clearly,  $v_a(b+c) = v_a(b)v_a(c)$  for any  $b, c \in \mathbb{F}^{r+1}$ . Let us show that  $v_a$  is indeed an eigenvector and en-route also compute its eigenvalue.

$$(Mv_a)(b) = \sum_{c \in \mathbb{F}^{r+1}} M_{bc} \cdot v_a(c)$$

<sup>3</sup>For example, if we view the elements of  $\mathbb{F}$  as element of  $\{0, 1\}^t$  then we can define  $L(a_0, a_1, \dots, a_{t-1}) = a_0$ .

$$\begin{aligned}
&= \sum_{x,y \in \mathbb{F}} v_a(b + y(1, x, \dots, x^r)) \\
&= \left( \sum_{x,y \in \mathbb{F}} v_a(y, yx, \dots, yx^r) \right) \cdot v_a(b).
\end{aligned}$$

Therefore  $\lambda_a = \sum_{x,y \in \mathbb{F}} v_a(y, yx, \dots, yx^r)$  is the eigenvalue of  $v_a$ . Set  $p_a(x) = \sum_{i=0}^r a_i x^i$  and write

$$\begin{aligned}
\lambda_a &= \sum_{x,y \in \mathbb{F}} v_a(y, yx, \dots, yx^r) = \sum_{x,y \in \mathbb{F}} (-1)^{L(y \cdot p_a(x))} \\
&= \sum_{\{x,y \in \mathbb{F} : p_a(x)=0\}} (-1)^{L(y \cdot p_a(x))} + \\
&\quad \sum_{\{x,y \in \mathbb{F} : p_a(x) \neq 0\}} (-1)^{L(y \cdot p_a(x))}.
\end{aligned}$$

If  $p_a(x) = 0$ , then  $(-1)^{L(y \cdot p_a(x))} = 1$  for all  $y$ , thus such an  $x$  contributes  $q$  to  $\lambda_a$ . If  $p_a(x) \neq 0$  then  $y \cdot p_a(x)$  takes on all values in  $\mathbb{F}$  as  $y$  varies, and hence  $(-1)^{L(y \cdot p_a(x))}$  varies uniformly over  $\{-1, 1\}$  implying that these  $x$ 's contribute nothing to  $\lambda_a$ . Therefore, when  $a = 0^n$  we have  $\lambda_a = q^2$ . Otherwise, when  $a \neq 0^n$ ,  $p_a$  has at most  $r$  roots, and therefore  $\lambda_a \leq rq$ .  $\blacksquare$

**Proof of left inequality of Theorem 1.1:** Let  $A$  be the adjacency matrix of  $G$  and note that as  $A$  is symmetric we have  $\lambda_2 = \max_{0 \neq x \perp 1^n} \langle xA, x \rangle / \langle x, x \rangle$ . For a set  $S \subseteq V(G)$  let  $x_S$  be the vector satisfying  $x_i = 1$  when  $i \in S$  and  $x_i = 0$  otherwise, and note that  $\langle x_S A, x_S \rangle = 2e(S)$  and  $\langle x_S A, x_{\bar{S}} \rangle = e(S, \bar{S})$ . Set  $x = |\bar{S}| \cdot x_S - |S| \cdot x_{\bar{S}}$  and note that  $x \perp 1^n$ . Therefore,

$$\langle xA, x \rangle \leq \lambda_2 \langle x, x \rangle = \lambda_2 (|S| + |\bar{S}|) |S| |\bar{S}|,$$

and

$$\langle xA, x \rangle = 2|S|^2 e(\bar{S}) + 2|\bar{S}|^2 e(S) - 2|S| |\bar{S}| e(S, \bar{S}),$$

giving

$$(2.1) \quad e(S, \bar{S}) \geq \frac{|S| e(\bar{S})}{|\bar{S}|} + \frac{|\bar{S}| e(S)}{|S|} - \frac{\lambda_2 (|S| + |\bar{S}|)}{|S| |\bar{S}|}.$$

As  $G$  is  $d$ -regular we have  $e(S) = \frac{1}{2}(d|S| - e(S, \bar{S}))$  and  $e(\bar{S}) = \frac{1}{2}(d|\bar{S}| - e(S, \bar{S}))$ . Plugging this into (2.1) and using  $|S| \leq n/2$ , we complete the proof by inferring that

$$e(S, \bar{S}) \geq (d - \lambda_2) |S| |\bar{S}| / n \geq \frac{1}{2} (d - \lambda_2) |S|.$$

### 3 Concluding Remarks

**Variants of Theorem 1.4:** Let us first show how to construct an expander with  $\Theta(n)$  vertices. Given  $n$  let  $q = 2^t$  and  $q^4/40 \leq r_0 \leq q^4/2$  be such that

$n/q^4 \leq q^{4r_0+12} \leq n$  (the existence of such  $q$  and  $r_0$  was proved as part of Theorem 1.4). We start by using Theorem 1.4 to construct a  $[q^{4r_0+12}, 12, \delta]$ -expander  $E$  satisfying  $n/q^4 \leq q^{4r_0+12} \leq n$ . If  $n/32 \leq q^{4r_0+12}$  we return  $E$ . Otherwise set  $t = \lfloor n/16q^{4r_0+12} \rfloor < q^4$  and use exhaustive search to find a 6-regular expander  $E'$  on  $12t$  vertices (which exists by Theorem 1.2). This step takes time  $q^{O(q^4)}$ , which is polynomial in the size of  $E$  because  $|E| \geq q^{\frac{1}{10}q^4}$  as  $r \geq q^4/40$ . We now replace every edge of  $E$  with  $t$  parallel edges to get a  $[q^{4r_0+12}, 12t, \delta]$ -expander  $E''$ . We then define  $E'' \circ E'$  to be the final 12-regular graph on  $m$  vertices with  $n/2 \leq m \leq n$ .

We now show that for every  $t$  we can construct a fully explicit  $[2^{t \lfloor 2^t/t \rfloor}, d, \delta]$ -expander for some constants  $d, \delta > 0$ . Thus, for every  $n$  we can construct such an expander of size  $n \leq m \leq n^2$ . We use the previous argument to find an expander of size  $2^{2t} \leq m \leq c2^{2t}$ . As noted in Section 1 we can then turn it into a constant degree expander  $E_1$  of size precisely  $2^{2t}$ . This step takes time  $2^{O(t)}$ . It is useful to “name” the vertices of  $E_1$  using pairs of elements of  $\mathbb{F}_{2^t}$ . Set  $E_2 = LD(2^t, \lfloor 2^t/t \rfloor - 3)$  and define  $E_3 = E_2 \circ E_1$  as the final constant degree expander on  $2^{t \lfloor 2^t/t \rfloor}$  vertices. To see that  $E_3$  is fully explicit, note that we can view a vertex of  $LD(q, r)$  as composed of  $r + 1$  elements of  $\mathbb{F}_q$ . Therefore, a vertex of  $E_3 = E_2 \circ E_1$  can be viewed as  $r + 1 = \lfloor 2^t/t \rfloor - 2$  elements  $(a_0, \dots, a_r)$  of  $\mathbb{F}_{2^t}$  (representing a vertex of  $E_2$ ) and another pair of elements  $x, y$  of  $\mathbb{F}_{2^t}$  (representing a vertex of  $E_1$ ). Suppose the degree of  $E_1$  is  $d'$  in which case the degree of  $E_3$  is  $2d'$ . Given  $r + 3$  elements  $(a_0, \dots, a_r, x, y)$  of  $\mathbb{F}_{2^t}$  and  $i \in [2d']$  we do the following. If  $1 \leq i \leq d'$  we return  $(a_0, \dots, a_r, x', y')$ , where  $(x', y')$  is the  $i^{\text{th}}$  neighbor of vertex  $(x, y)$  in  $E_1$ . We can do so by generating  $E_1$  from scratch in time  $2^{O(t)}$ . If  $d' + 1 \leq i \leq 2d'$ , we return the vertex  $(a'_0, \dots, a'_r, x, y)$ , where  $a'_i = a_i + yx^i$ . To do so we use a representation of  $\mathbb{F}_{2^t}$  that we find using exhaustive search in time  $2^{O(t)}$ . We finally note that one can easily adopt our arguments to get *space efficient* variants of our constructions. We omit the details.

**Edge expansion close to  $\frac{1}{2}$ :** The expanders we constructed have a positive edge expansion. However, by applying Theorem 1.1 it is easy to see that for every  $\epsilon$  we can raise the graphs we construct to an appropriate power to get edge-expansion  $\frac{1}{2} - \epsilon$ . In fact, to get edge-expansion  $\frac{1}{2} - \epsilon$  one needs the degree to be *poly*( $1/\epsilon$ ).

**Eigenvalue gap:** As we have mentioned before all the previous constructions of bounded-degree expanders did so via constructing a graph, whose second eigenvalue is bounded away from  $d$ . Theorem 1.1 implies that if  $G$  is an  $[n, d, \delta]$ -expander then its second largest eigenvalue

is at most  $d(1 - \frac{1}{2}\delta^2)$ . As we can construct expanders with edge expansion close to  $\frac{1}{2}$ , these graphs have second largest eigenvalue at most roughly  $\frac{7}{8}d$ . By adding loops and raising the resulting graphs to an appropriate power one can get expanders in which all eigenvalues are, in absolute value, at most some fractional power of the degree of regularity.

**Expanders with smaller degree:** The expanders we construct have constant degree larger than 3. In order to get 3-regular expander one can take any constant degree  $d$ -regular expander and apply a replacement product with a cycle of length  $d$ . Definition 1.1 implies that the new degree is 4, but it is easy to see that when  $d$  is a constant we do not have to duplicate each edge of the “large” graph  $d$  times, as keeping a single edge guarantees a positive expansion. This way we can get a 3-regular expander, which is clearly the smallest possible degree of regularity.

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## Appendix

**Proof of right inequality of Theorem 1.1:** Let  $Q = dI - A$  be the Laplace matrix of  $G$ . Our goal is to prove that all but one of the eigenvalues of  $Q$  are at least  $\frac{1}{2}\delta^2d$ . Let  $z = (z_1, z_2, \dots, z_n)$  be an eigenvector of  $Q$  with the smallest nontrivial eigenvalue  $\lambda$ , where  $V(G) = \{1, 2, \dots, n\}$ . Recall that for every set  $U$  of at most half the vertices of  $G$  there are at least  $c|U|$  edges between  $U$  and its complement, where  $c = \delta d$  is some positive constant. Clearly  $\sum_i z_i = 0$ . Without loss of generality assume that  $m \leq n/2$  of the entries of  $z$  are positive (otherwise, replace  $z$  by  $-z$ ), and that  $z_1 \geq z_2 \geq \dots \geq z_m > 0 \geq z_{m+1} \geq \dots \geq z_n$ . Define  $x_i = z_i$  for  $i \leq m$ , and  $x_i = 0$  otherwise. Since  $x_j = 0$  for all  $j \geq n/2$ ,

$$\begin{aligned} \sum_{ij \in E} |x_i^2 - x_j^2| &= \sum_{ij \in E, i < j} (x_i^2 - x_j^2) \\ (3.2) \quad &\geq \sum_{ii < n/2} (x_i^2 - x_{i+1}^2)ci = c \sum_{i=1}^n x_i^2. \end{aligned}$$

Note that  $(Qz)_i = \lambda z_i$  for all  $i$  and hence  $\lambda = \frac{\sum_{i=1}^m (Qz)_i z_i}{\sum_{i=1}^m z_i^2}$ . However,

$$\begin{aligned} \sum_{i=1}^m (Qz)_i z_i &= \sum_{i=1}^m (dz_i^2 - \sum_{j, ij \in E} z_i z_j) \\ &= \sum_{i, j \leq m, ij \in E} (z_i - z_j)^2 \\ &\quad + \sum_{i \leq m, j > m, ij \in E} z_i(z_i - z_j) \\ &\geq \sum_{ij \in E} (x_i - x_j)^2. \end{aligned}$$

As  $\sum_{i=1}^m z_i^2 = \sum_{i=1}^n x_i^2$  we conclude, using Cauchy Schwartz (twice) that

$$\begin{aligned} \lambda &\geq \frac{\sum_{ij \in E} (x_i - x_j)^2}{\sum_{i=1}^n x_i^2} \\ &= \frac{\sum_{ij \in E} (x_i - x_j)^2 \sum_{ij \in E} (x_i + x_j)^2}{\sum_i x_i^2 \sum_{ij \in E} (x_i + x_j)^2} \\ &\geq \frac{(\sum_{ij \in E} |x_i^2 - x_j^2|)^2}{\sum_i x_i^2 2d \sum_i x_i^2} \geq \frac{c^2}{2d}, \end{aligned}$$

where the last inequality follows from (3.2). Therefore,  $\lambda \geq \frac{c^2}{2d} = \frac{1}{2}\delta^2d$ . ■