Contents

Chapter 1

Introduction

Quantum computation can be seen as the use of quantum mechanical phenomena to perform operation on data. It holds great potential due to several properties of quantum mechanics, like superposition and entanglement [1]. Harnessing these properties in a constructive way can give immense computation power. Research in this field is being done in countless locations including the top academic institutions and personals, but it is still a long way from realizing computation power comparable to that of a classical computer. In the mean time this research is expanding our knowledge about the basic physical laws governing our world, and giving us new insights on quantum mechanics, especially on the border line between quantum and classic. This can be seen, for example, in the mechanism of decoherence which is being investigated intensively [2]. In many other fields of physics there is a vivid exchange of ideas of ideas where between Quantum information and "normal" physics. One example is the use of various types of systems to violate Bell's inequalities and the positive effect this operation on the research involving those systems.

1.1 Qubits

The basic building blocks of a computer are bits which can take the value of 0 or 1. Their quantum counterparts are the qubits, which can take the value of 0, 1 or a superposition of those. A quantum system that has two levels can, in theory, be considered as a qubit. However in order to get some computation power, there are a few conditions that need to be satisfied [3]:

1. The system should be scalable.

2. It should be possible to initialize the system.

3. The decoherence times need to be long

4. A universal set of quantum gates can be realized

5. Measurements can be performed efficiently.

These conditions are far from simple to achieve. Many physical realizations of qubits have been suggested [4]. One can divide this suggestion to categories according to the field in physics to which they belong. The main categories are atomic, optic and solid state. Within each category there are many possible qubits proposals and further sub division is also common. Each proposal has its on advantages and disadvantages and up till now there is not an obvious "winner".

The category of solid state has the advantages that it is scaleable and offer various ways of performing measurements on the qubits. The main disadvantage is the strong interaction with the environment which cause decoherence. Within solid state qubits the two main categories are quantum dots, which exploit the properties of semiconductors, and superconducting qubits which exploit superconductivity.

1.2   Superconductivity and SCB

The quantum phenomenon of Superconductivity, as candidate to be harnessed for calculation, offers some remarkable features. Its source is purely quantum mechanical but it is much simpler to observe than most other quantum effects, like the spin of an electron, the structure of an atom, or detection of smaller particles. The current theory for superconductivity is BCS [5] which is based on the bounding of electrons in pairs, known as Cooper Pairs. The quantum phenomenon of Superconductivity is explained by the existence of a *macroscopic* number of Cooper pairs, in a *single quantum state*. This implies imply that it provides a superb opportunity to connect our macroscopic world to the peculiar nature of quantum mechanics.

The single Cooper pair box (SCB) is a quantum system that can be described by the number of Cooper pairs in a superconducting island [6] . Some reference number of cooper pairs can be defined so the surplus beyond it would be considered. If one limits ones interest to the states of 0 or 1 extra Cooper pairs this system can be considered as a qubit.

Chapter 2

Investigating The Quantum Zeno Effect Using Continuous Measurement

2.1 Introduction

The effect of measurements on an unstable quantum mechanical system can be either the suppression or acceleration of the decay rate. The effect of suppression was first studied by Misra and Sudarshan [7], who considered projective measurements repeated with interval T_R, and coined the name of the quantum Zeno effect. A necessary condition for the effect to occur is that T_R is smaller than the so called jump time [8], before which the probability of remaining in the original state decrease approximately quadratically $\left(P \sim 1 - V t^2\right)$. For times much longer than the jump time the probability decrease exponentially $\left(P \sim e^{-\Gamma t}\right)$, according to Fermi's golden rule. The deviation from the exponential behavior was observed experimentally by Wilkinson et al [9]. If the repetition interval is longer than the jump time but still not in the exponential regime, an acceleration of the decay might occur [10, 11, 12] which is called anti Zeno effect (AZE).

In addition to the decay process, QZE and AZE can also affect coherent oscillations [13]. Although this distinction is not always made, there are fundamental differences between the processes. Coherent oscillations are the evolution of a closed quantum system. Such a system, if it is in an eigenstate of its Hamiltonian, will remain in that state. Otherwise it will oscillate coherently between the different states [14]. The decay process, on the other hand, is caused by coupling to a continuum of states, or an environment. This coupling induce a type of evolution which is irreversible and act as a source of decoherence [2].

The basic notion of QZE, or AZE, can be understood differently for the two types of processes. For the decay process, the suppression, or acceleration of the

decay rate can be discussed clearly [10, 11, 13] since the decay rate in the absence of measurement is defined. When the coherent oscillations are affected, usually the concept of localization, the breaking of the superpositions, is discussed in literature[12].

Since the coherent evolution of a closed system without measurement is known, one can derive how long the quadratic behavior persists, thus the jump time is known. For the decaying process this is not possible in general, but it is believed the jump time is quite short and that the exponential behavior is dominant rather early. This means that QZE is harder to observe for such process but also that it is more likely to observe AZE. If one considers a system which has only one type of evolution there is no conflict, but if a system which has both types, the measurement can affect both and it should be specified which effect is interesting. The coherent evolution is easier to be affected, so whenever it's included it will be dominant. Thus, if the decay process is the target, the coherent evolution should not be affected. This require QND type measurements where the observable commutes with the Hamiltonian of the closed system [15].

An important example of decay process is errors in quantum computation which are caused by decoherence. The prospect of preventing errors, or decoherence using QZE is widely studied [16, 17] and it holds one of the promises for realizable quantum computation. Therefore it is of great importance to investigate QZE for decay process, especially in the context of qubits, and to confirm it's applicability through experiments. This is the purpose of this chapter.

In Ref. [7] and many papers thereafter, the projection postulate was used to describe the back-action of the measurement. This method is appropriate when the time needed to perform individual projective measurement is much smaller than any other time scale in the system, especially the jump time. When the time scales are comparable another description of the back-action is needed, which have to be based on the way the measurement is performed. In atomic physics systems, the measurement is done typically by detecting the photon emitted in the decay, and so it is indirect. Recent development in circuit cavity quantum electrodynamics (CCQED) [18, 19] made it possible to engineer systems which simulate the physics of atoms interacting with quantized electromagnetic radiation in electrical circuits. In these systems it is possible to perform a direct QND measurement on the "atom". Those measurements are weak, so the back-action they induce is pure dephasing [20], i.e. the exponential decay of the off diagonal elements in the density matrix. In this chapter, we model the back-action as pure dephasing induced on the qubit.

Other models which go beyond the projection postulate include treating the measurements as time dependent modulation/perturbation [16], analyzing the interaction between the unstable system and another quantum mechanical system which operates as the measurement apparatus [13] and considering the effect of general measurements on the reduced density matrix [21].

Most experimental observation of QZE [22] were performed for coherent oscillations. Recently, a few attempts to observe QZE and AZE in truly decaying systems [23] were done. In the developing field of CCQED new regimes are becoming experimentally accessible. In this paper we present a realizable setup
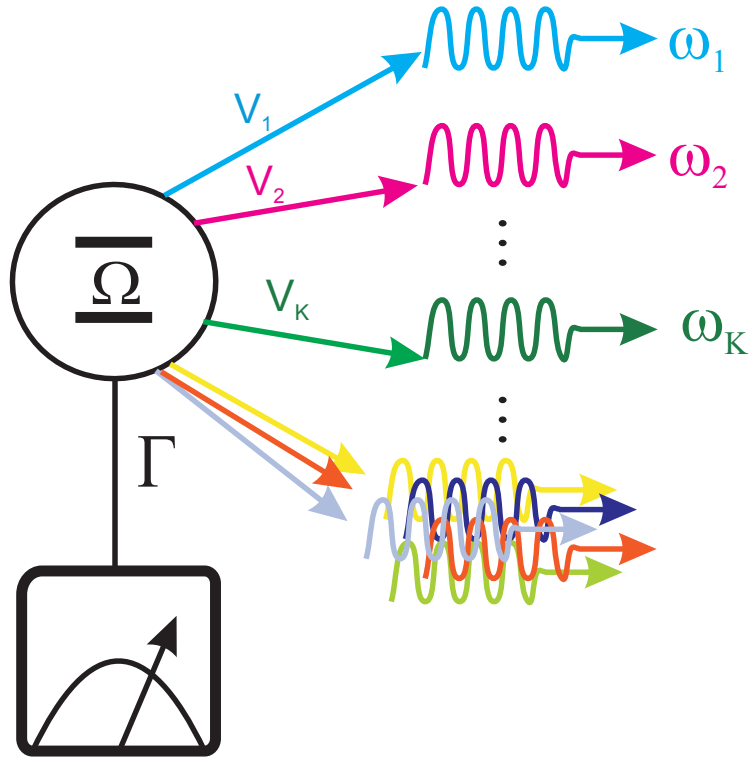
Figure 2.1: An illustration of the model. A two level system (qubit) with energy splitting Ω can decay to a continuum of modes. The coupling to a mode ω_k is V_k. The qubit is also independently measured in a rate Γ_D.

where QZE and AZE can be observed.

2.2 The model: measurement of a decaying system

2.2.1 The decaying system

An illustration of our model is given in fig. 2.1. We explicitly consider a two-level system (qubit) transversely coupled to a bath. The Hamiltonian is given by

$$
\begin{aligned}
H &= H_q + H_B + H_I, \\
H_q &= \frac{\hbar\Omega}{2}\sigma_z,
\end{aligned}
$$

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$$H_B = \hbar \sum_k \omega_k B_k^\dagger B_k,$$

$$H_I = \hbar \sum_k V_k \sigma_x \left(B_k + B_k^\dagger \right). \qquad (2.1)$$

Here H_q, H_B and H_I are the Hamiltonians of the qubit, the bath and their interaction respectively, Ω, ω_k and V_K are the frequencies of the qubit, the bath mode k and their interaction respectively, σ_i is the Pauli i-matrix and $B_k^\dagger (B_k)$ is the creation (annihilation) operator.

Since $\sigma_x = \sigma_+ + \sigma_-$ this is the term causing decay and excitation. The rotating wave approximation (RWA) means we neglect the terms including $\sigma_- B_k$ and $\sigma_+ B_k^\dagger$. We can write the interaction Hamiltonian as:

$$H_I = \hbar \sum_k \left(V_k \sigma_+ B_k + V_k^* B_k^\dagger \sigma_- \right). \qquad (2.2)$$

This form implies that $N \equiv \sigma_z + \sum_k B_k^\dagger B_k$, the total number of excitations, is conserved. Thus the combined Hilbert space is factorized to subspaces corresponding to different values of N, which do not interact with each other. We will consider just the sub space N = 1. The set of states $\{|e, 0\rangle, |g, k\rangle\}$ span this subspace, where $|e, 0\rangle$ means the qubit is excited and all bath modes are in the ground state and $|g, k\rangle$ means the qubit is in the ground state and the bath mode k is excited. So we can write the Hamiltonian as

$$
\begin{aligned}
H &= \hbar\Omega \left(|e, 0\rangle \langle e, 0| - 1/2 \right) + \hbar \sum_k \omega_k |g, k\rangle \langle g, k| \\
&+ \hbar \sum_k \left(V_k |g, k\rangle \langle e, 0| + V_k^* |e, 0\rangle \langle g, k| \right). \qquad (2.3)
\end{aligned}
$$

2.2.2 The Lindblad master equation to model measurement

The measurements induce decoherence on the system [2]. Such process can be described by the Lindblad master equation [24]:

$$\dot{\rho}_S = \frac{1}{i\hbar}[H_S, \rho_S] + \sum_k \left[L_k \rho_S L_k^\dagger - \frac{1}{2}\{\rho_S, L_k^\dagger L_k\} \right], \qquad (2.4)$$

where ρ_S is the reduced density matrix, H_S is the Hamiltonian for the system without the measurement and L_k are the Lindblad operators.

The minimal effect possible is pure dephasing. In some measurement it is the dominant effect [19, 18]. To describe this effect we take the Lindblad operator to be:

$$L_1 = \sqrt{\frac{\Gamma}{2}}\sigma_z \otimes I_B,$$
$$L_{i \neq 1} = 0. \tag{2.5}$$

This operator is chosen so that in the absence of the bath we get an exponential decay with rate Γ for the off diagonal elements in the density matrix of the qubit alone. It is important to note that we did not trace over the bath, so the equation above for the reduced density matrix describe the combined qubit + bath system. This is the reason for the explicit tensor product with a unity operator (I_B) acting on the bath.

2.3 Evolution of the system

Using Eq. (2.1 ,2.4, 2.5) we can write the complete equation of motion for the density matrix:

$$\dot{\rho} = \frac{1}{i\hbar}[H_0, \rho] + \frac{1}{i\hbar}[H_I, \rho] + \frac{\Gamma}{2}\left(\sigma_z \rho \sigma_z - \rho\right), \tag{2.6}$$

where

$$H_0 = H_q + H_B$$

and we dropped the index S on the density matrix.

2.3.1 Superoperators

It is cumbersome to deal with a equation in the form of 2.6 , where we have operators acting both from the right and from the left. This is the motivation to introduce super operators:

$$L^0\{\rho\} \equiv \frac{1}{i\hbar}[H_0, \rho] + \frac{\Gamma}{2}\left(\sigma_z \rho \sigma_z - \rho\right),$$
$$L^I\{\rho\} \equiv \frac{1}{i\hbar}[H_I, \rho].$$

If we consider the density matrix as rank 2 tensor (two indices), the super operator is a rank 4 tensor (4 indices) and the operation includes two summations instead of just one in the "normal" matrix and vector notation. So the operation of a super operator will be as:

$$\rho'_{jk} = L_{jklm}\rho_{lm}, \tag{2.7}$$

where there's implicit summation is over repeated indices.

We can also multiply two super operators

$$L3_{jklm} = L1_{jknp}L2_{nplm}. \qquad (2.8)$$

In our system, we can write the density matrix as:

$$\rho = \rho_{00}\,|e,0\rangle\langle e,0| + \sum_{k\neq 0}\rho_{k0}\,|g,k\rangle\langle e,0|$$
$$+ \sum_{k\neq 0}\rho_{0k}\,|e,0\rangle\langle g,k| + \sum_{k\neq 0,j\neq 0}\rho_{jk}\,|g,j\rangle\langle g,k|. \qquad (2.9)$$

We can now look at the operation of the super operator:

$$L^0\{\rho\} \equiv \frac{1}{i\hbar}[H_0,\rho] + \frac{\Gamma}{2}\left(\sigma_z\rho\sigma_z - \rho\right)$$

$$= -i\left(\sum_{k\neq 0}(\omega_k - \Omega)\,\rho_{k0}\,|g,k\rangle\langle e,0| + \sum_{k\neq 0}(\Omega - \omega_k)\,\rho_{0k}\,|e,0\rangle\langle g,k|\right.$$

$$\left. + \sum_{k\neq 0,j\neq 0}(\omega_j - \omega_k)\,\rho_{jk}\,|g,j\rangle\langle g,k|\right)$$

$$- \Gamma\left(\sum_{k\neq 0}\rho_{k0}\,|g,k\rangle\langle e,0| + \sum_{k\neq 0}\rho_{0k}\,|e,0\rangle\langle g,k|\right). \qquad (2.10)$$

We can see that this operator gives all the off diagonal elements a rotating phase caused by unitary evolution, and that the element that are off diagonal in the qubit sub space decay with rate Γ which represents dephasing.

We can write this operator in tensor notation:

$$L^0_{0000} = 0,$$
$$L^0_{j0k0} = \delta_{jk}\left(i\left(\omega_k - \Omega\right) - \Gamma\right),$$
$$L^0_{0j0k} = \delta_{jk}\left(i\left(\Omega - \omega_k\right) - \Gamma\right),$$
$$L^0_{jklm} = \delta_{jl}\delta_{km}i\left(\omega_j - \omega_k\right)$$

and all other elements vanish.

The operation of L^I can also be calculated:

$$iL^I\{\rho\} \equiv \frac{1}{\hbar}[H_I,\rho_S]$$

$$= \rho_{00}\sum_{k\neq 0}\left(V_k^*\,|g,k\rangle\langle e,0| - V_k\,|e,0\rangle\langle g,k|\right)$$

$$+ \sum_{k\neq 0}\rho_{k0}\sum_{j\neq 0}\left(V_j\delta_{jk}\,|e,0\rangle\langle e,0| - V_j\,|g,j\rangle\langle g,k|\right)$$

$$+ \sum_{k\neq 0}\rho_{0k}\sum_{j\neq 0}\left(V_j^*\,|g,j\rangle\langle g,k| - V_j^*\delta_{kj}\,|e,0\rangle\langle e,0|\right)$$

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$$+ \sum_{k\neq 0, j\neq 0} \rho_{jk} \sum_{l\neq 0} \left(V_l \delta_{lj} \ket{e,0}\bra{g,k} - V_l^* \delta_{lk} \ket{g,j}\bra{e,0} \right)$$

$$= \ket{e,0}\bra{e,0} \sum_{k\neq 0} \left(V_k \rho_{k0} - V_k^* \rho_{0k} \right)$$

$$+ \sum_{k\neq 0} \ket{e,0}\bra{g,k} \left(\sum_{j\neq 0} V_j \rho_{jk} - V_k \rho_{00} \right)$$

$$+ \sum_{k\neq 0} \ket{g,k}\bra{e,0} \left(V_k^* \rho_{00} - \sum_{j\neq 0} V_j \rho_{kj} \right)$$

$$+ \sum_{k\neq 0, j\neq 0} \ket{g,j}\bra{g,k} \left(V_j^* \rho_{0k} - V_j \rho_{k0} \right).$$

We can write it in tensor notation:

$$
\begin{aligned}
L^I_{00k0} &= V_k, \\
L^I_{000k} &= -V_k^*, \\
L^I_{k000} &= V_k^*, \\
L^I_{k0jl} &= -V_l^* \delta_{jk}, \\
L^I_{0k00} &= -V_k, \\
L^I_{0kjl} &= V_j^* \delta_{lk}, \\
L^I_{kjl0} &= -V_k \delta_{lj}, \\
L^I_{kj0l} &= V_k^* \delta_{jl}
\end{aligned}
$$

and all other elements vanish.

Since the structure of this tensor is determined by the difference between cases where indices take the value 0 or not, we can divide it to the 16 different cases. We can write the 16 tensors in a matrix: This can be represented as a matrix:

$$L^I_{\alpha,\beta,\gamma,\sigma} = \begin{pmatrix} 0 & V_k & -V_k^* & 0 \\ V_k^* & 0 & 0 & -V_l^* \delta_{jk} \\ -V_k & 0 & 0 & V_j^* \delta_{lk} \\ 0 & -V_k \delta_{lj} & V_k^* \delta_{jl} & 0 \end{pmatrix}. \tag{2.11}$$

Here the first row (column) is for the case where the first (last) two indices are 00, the second row (column) is for the case where the first (last) two indices are 0k, the third row (column) is for the case where the first (last) two indices are k0 and the forth row (column) is for the case where the first (last) two indices are kj.

The way tensor are multiplied imply that doing a matrix multiplication when each term is a tensor is equivalent.

2.3.2 Perturbation Theory

Our equation of motion for the density matrix can now be written using the super operators:

$$\dot{\rho} = \left(L^0 + L^I\right)\rho. \tag{2.12}$$

We can write the solution as:

$$\rho(t) = e^{L^0 t}\left(1 + \int_0^t \tilde{L}(t_1)dt_1 + \int_0^t \int_0^{t_1} \tilde{L}(t_1)\tilde{L}(t_2)dt_2 dt_1 + ...\right)\rho(0) \tag{2.13}$$

where

$$\tilde{L}(t) = e^{-L^0 t}L^I e^{L^0 t}. \tag{2.14}$$

To verify we can simply calculate the time derivative:

$$
\begin{aligned}
\dot{\rho}(t) &= L^0 e^{L^0 t}\left(1 + \int_0^t \tilde{L}(t_1)dt_1 + \int_0^t \int_0^{t_1} \tilde{L}(t_1)\tilde{L}(t_2)dt_2 dt_1 + ...\right)\rho(0) \\
&+ e^{L^0 t}\left(\tilde{L}(t) + \tilde{L}(t)\int_0^t \tilde{L}(t_2)dt_2 + ...\right)\rho(0) \\
&= (L^0 + L^I)\rho(t). \tag{2.15}
\end{aligned}
$$

The exponent of a super operator is defined, like the exponent of a matrix, by the power series. Since L^0 is diagonal we can get a simple expression for $U(t) \equiv e^{L^0 t}$. We can see that by looking, as an example, at the second term in the series, $L^0 L^0 t^2$:

$$
\begin{aligned}
\left(L^0 L^0 t^2\right)_{0000} &= L^0_{00jk}L^0_{jk00} = 0, \\
\left(L^0 L^0 t^2\right)_{j0k0} &= L^0_{j0lm}L^0_{lmk0}t^2 L^0_{j0l0}L^0_{l0k0}t^2 \\
&= \delta_{jl}\left(i\left(\omega_l - \Omega\right) - \Gamma\right)\delta_{lk}\left(i\left(\omega_k - \Omega\right) - \Gamma\right)t^2 \\
&= \delta_{kj}\left(i\left(\omega_k - \Omega\right) - \Gamma\right)^2 t^2, \\
\left(L^0 L^0 t^2\right)_{0j0k} &= L^0_{0jlm}L^0_{lm0k}t^2 = L^0_{0j0l}L^0_{0l0k}t^2 \\
&= \delta_{jl}\left(-i\left(\omega_l - \Omega\right) - \Gamma\right)\delta_{lk}\left(-i\left(\omega_k - \Omega\right) - \Gamma\right)t^2 \\
&= \delta_{kj}\left(-i\left(\omega_k - \Omega\right) - \Gamma\right)^2 t^2, \\
\left(L^0 L^0 t^2\right)_{jklm} &= L^0_{jknp}L^0_{nplm}t^2 = \delta_{jn}\delta_{kp}i\left(\omega_j - \omega_k\right)\delta_{nl}\delta_{pm}i\left(\omega_n - \omega_p\right)t^2 \\
&= -\delta_{jl}\delta_{km}\left(\omega_j - \omega_k\right)^2 t^2
\end{aligned}
$$

and all other component vanish.

The sum will give an exponent in the diagonal elements:

$$
\begin{aligned}
U(t)_{0000} &= 1, \\
U(t)_{j0k0} &= \delta_{jk}e^{i(\omega_k - \Omega)t - \Gamma t}, \\
U(t)_{0j0k} &= \delta_{jk}e^{i(\Omega - \omega_k)t - \Gamma t}, \\
U(t)_{jklm} &= \delta_{jl}\delta_{km}e^{i(\omega_j - \omega_k)t}
\end{aligned}
$$

and all other elements vanish.

Using $U(t)$, we can calculate $\tilde{L}(t)$:

$$
\begin{aligned}
\tilde{L}(t)_{00k0} &= \left(U(-t)L^I U(t)\right)_{00k0} = U(-t)_{00jl}L^I_{jlmn}U(t)_{mnk0} \\
&= U(-t)_{0000}L^I_{00m0}U(t)_{m0k0} = V_m \delta_{mk}e^{i(\omega_k-\Omega)t-\Gamma t} = V_k e^{i(\omega_k-\Omega)t-\Gamma t}, \\
\tilde{L}(t)_{000k} &= \left(U(-t)L^I U(t)\right)_{000k} = U(-t)_{00jl}L^I_{jlmn}U(t)_{mn0k} \\
&= U(-t)_{0000}L^I_{000n}U(t)_{0n0k} = -V^*_n \delta_{nk}e^{-i(\omega_k-\Omega)t-\Gamma t} \\
&= -V^*_k e^{-i(\omega_k-\Omega)t-\Gamma t}, \\
\tilde{L}(t)_{k000} &= \left(U(-t)L^I U(t)\right)_{k000} = U(-t)_{k0jl}L^I_{jlmn}U(t)_{mn00} \\
&= U(-t)_{k0j0}L^I_{j000}U(t)_{0000} = V^*_j \delta_{jk}e^{-i(\omega_k-\Omega)t+\Gamma t} = V^*_k e^{-i(\omega_k-\Omega)t+\Gamma t}, \\
\tilde{L}(t)_{k0jl} &= \left(U(-t)L^I U(t)\right)_{k0jl} = U(-t)_{k0mn}L^I_{mnpr}U(t)_{prjl} \\
&= U(-t)_{k0m0}L^I_{m0pr}U(t)_{prjl} \\
&= -\delta_{km}e^{-i(\omega_m-\Omega)t+\Gamma t}V^*_r \delta_{mp}\delta_{pj}\delta_{rl}e^{i(\omega_p-\omega_r)t} \\
&= -e^{-i(\omega_l-\Omega)t+\Gamma t}V^*_l \delta_{kj}, \\
\tilde{L}(t)_{0k00} &= \left(U(-t)L^I U(t)\right)_{0k00} = U(-t)_{0kjl}L^I = \\
&= -V_j \delta_{kj}e^{-i(\omega_j-\Omega)t+\Gamma t} = -V_k e^{-i(\omega_k-\Omega)t+\Gamma t}, \\
\tilde{L}(t)_{0kjl} &= \left(U(-t)L^I U(t)\right)_{0kjl} = U(-t)_{0kmn}L^I_{mnpr}U(t)_{prjl} \\
&= U(-t)_{0k0n}L^I_{0npr}U(t)_{prjl} = \delta_{kn}e^{i(\omega_k-\Omega)t+\Gamma t}V^*_p \delta_{nr}\delta_{pj}\delta_{rl}e^{i(\omega_p-\omega_r)t} \\
&= e^{i(\omega_j-\Omega)t+\Gamma t}V^*_j \delta_{kl}, \\
\tilde{L}(t)_{kjl0} &= \left(U(-t)L^I U(t)\right)_{kjl0} = U(-t)_{kjmn}L^I_{mnpr}U(t)_{prl0} \\
&= U(-t)_{kjmn}L^I_{mnp0}U(t)_{p0l0} \\
&= -\delta_{km}\delta_{jn}e^{i(\omega_k-\omega_j)t}V_m \delta_{np}\delta_{pl}e^{i(\omega_l-\Omega)t-\Gamma t} \\
&= -V_k \delta_{jl}e^{i(\omega_k-\Omega)t-\Gamma t}, \\
\tilde{L}(t)_{kj0l} &= \left(U(-t)L^I U(t)\right)_{kj0l} = U(-t)_{kjmn}L^I_{mnpr}U(t)_{pr0l} \\
&= U(-t)_{kjmn}L^I_{mn0r}U(t)_{0r0l} \\
&= \delta_{km}\delta_{jn}e^{-i(\omega_k-\omega_j)t}V^*_m \delta_{nr}\delta_{rl}e^{i(\Omega-\omega_l)t-\Gamma t} \\
&= V^*_k \delta_{jl}e^{i(\Omega-\omega_k)t-\Gamma t}
\end{aligned}
$$

and all other elements vanish.

It might help getting an intuitive picture to arrange this elements in a matrix:

$$
\tilde{L} = \begin{pmatrix}
0 & \tilde{L}_{00k0} & \tilde{L}_{000k} & 0 \\
\tilde{L}_{k000} & 0 & 0 & \tilde{L}_{k0jl} \\
\tilde{L}_{0k00} & 0 & 0 & \tilde{L}_{0kjl} \\
0 & \tilde{L}_{kjl0} & \tilde{L}_{kj0l} & 0
\end{pmatrix}.
$$

We are interested in the probability of the qubit to decay after time t. This is straight forward to calculate using this formalism. We put the initial matrix

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in the pure undecayed state:

$$\rho(0) = |e,0\rangle \langle e,0|, \tag{2.16}$$

and the probability of no decay is given by the matrix element

$$P_{nd}(t) = \rho(t)_{00}. \tag{2.17}$$

Since we consider the interaction with the bath to be weak on the time scales we consider $(1 \gg Vt)$ we'll take the first non-vanishing order of perturbation:

$$
\begin{aligned}
P_{nd}(t) &= \rho(t)_{00} \simeq 1 - \int_0^t \int_0^{t_1} \left(\tilde{L}(t_1)\tilde{L}(t_2) \right)_{0000} dt_2 dt_1, \\
\left(\tilde{L}(t_1)\tilde{L}(t_2) \right)_{0000} &= \tilde{L}(t_1)_{00jk}\tilde{L}(t_2)_{jk00} \\
&= \tilde{L}(t_1)_{00k0}\tilde{L}(t_2)_{k000} + \tilde{L}(t_1)_{000k}\tilde{L}(t_2)_{0k00} \\
&= V_k e^{i(\omega_k - \Omega)t_1 - \Gamma t_1} V_k^* e^{-i(\omega_k - \Omega)t_2 + \Gamma t_2} \\
&\quad + V_k^* e^{-i(\omega_k - \Omega)t_1 - \Gamma t_1} V_k e^{-i(\omega_k - \Omega)t_2 + \Gamma t_2} \\
&= V_k V_k^* e^{\Gamma(t_2 - t_1)} \left(e^{i(\omega_k - \Omega)(t_1 - t_2)} + e^{i(\omega_k - \Omega)(t_2 - t_1)} \right) \\
&= V_k V_k^* e^{-\Gamma(t_1 - t_2)} 2 \cos\left((\omega_k - \Omega)(t_2 - t_1) \right). \tag{2.18}
\end{aligned}
$$

The Einstein summation rule for the index k means we need to sum over all bath modes, but before that, we can do the time integration. Denoting $\omega_k' = \omega_k - \Omega$ we get:

$$
\int_0^t \int_0^{t_1} \left(\tilde{L}(t_1)\tilde{L}(t_2) \right)_{0000} dt_2 dt_1 = V_k V_k^* 2 \times
$$
$$
\frac{(\Gamma t + 1)\omega_k'^2 + \Gamma^2(\Gamma t - 1) + e^{-\Gamma t}\left((\Gamma^2 - \omega_k'^2)\cos(\omega_k' t) - 2\omega_k' \Gamma \sin(\omega_k' t) \right)}{\left(\omega_k'^2 + \Gamma^2 \right)^2}.
$$

To check the validity of this expression we can look at the case of no measurement, $\Gamma = 0$, we get:

$$
\begin{aligned}
\int_0^t \int_0^{t_1} \left(\tilde{L}(t_1)\tilde{L}(t_2) \right)_{0000} dt_2 dt_1 &= V_k V_k^* \frac{2\left(1 - \cos(\omega_k' t)\right)}{\omega_k'^2} \\
&= V_k V_k^* t^2 sinc^2(\omega_k' t/2),
\end{aligned}
$$

which gives the Fermi golden rule for long times.

Assuming $\Gamma t \gg 1$ the expression simplify considerably:

$$
\int_0^t \int_0^{t_1} \left(\tilde{L}(t_1)\tilde{L}(t_2) \right)_{0000} dt_2 dt_1 = V_k V_k^* \frac{2\Gamma t}{\omega_k'^2 + \Gamma^2}. \tag{2.19}
$$

By considering the measurement time to be $1/\Gamma$ the assumption above can be understood as looking into the time range when the measurement have been repeated many times.

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We considered $\Gamma t \gg 1$ and $1 \gg Vt$. This means that the time range in which QZE and AZE happens is $1/V \gg t \gg 1/\Gamma$. Whether this time range exists and how long it is determine the possibility of observing the effect in a specific set up.

The decay rate, Γ_d, can be defined as:

$$P_{nd}(t) = e^{-\Gamma_d t}, \tag{2.20}$$

so we can estimate it by

$$\Gamma_d = -\frac{\ln P_{nd}}{t} \simeq -\frac{\ln\left(1 - V_k V_k^* \frac{2\Gamma t}{\omega_k'^2 + \Gamma^2}\right)}{t} \simeq V_k V_k^* \frac{2\Gamma}{\omega_k'^2 + \Gamma^2}. \tag{2.21}$$

To do the summation over k we can define the bath form factor or spectral density:

$$G(\omega) = \sum_k |V_k|^2 \, \delta(\omega - \omega_k). \tag{2.22}$$

So we can write the expression for the decay rate as:

$$\Gamma_d = \int d\omega G(\omega) F(\omega), \tag{2.23}$$

where

$$F(\omega) = \frac{2\Gamma}{(\omega - \Omega)^2 + \Gamma^2}. \tag{2.24}$$

So we see that the decay rate is given by a convolution of the spectral density with a Lorentzian. Equations of the form 2.23 were widely derived [13, 21, 11, 16] where $G(\omega)$ is unknown and $F(\omega)$ depends on the model of measurement. The form of $F(\omega)$ derived above, is quite simple and is based on a universal property of measurement process. Without any information $G(\omega)$ very little can be said about the decay rate. We can note that for a flat form factor $G(\omega) = const$ we get again the Fermi golden rule.

2.3.3 Consider Lorentzian as the form factor

An interesting example of the spectral density, which was considered extensively in literature is a Lorentzian as well,

$$G(\omega) = \gamma \frac{\kappa/2\pi}{\kappa^2 + (\omega - \omega_0)^2}, \tag{2.25}$$

where κ is the line width, ω_0 is the resonance frequency and γ is a normalization factor with units of frequency squared.
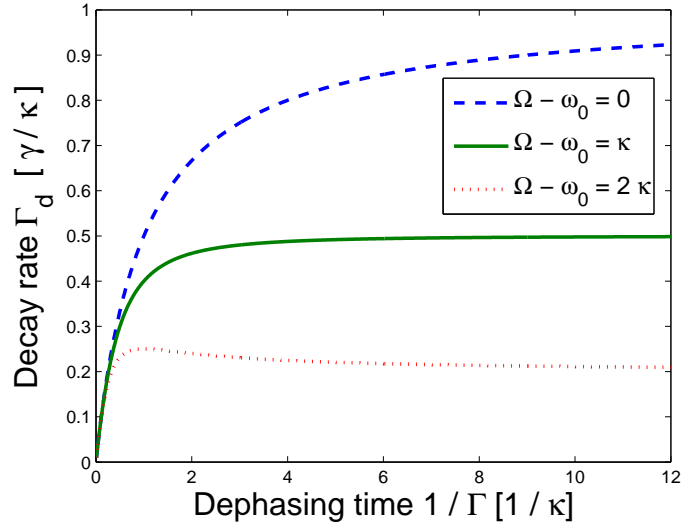
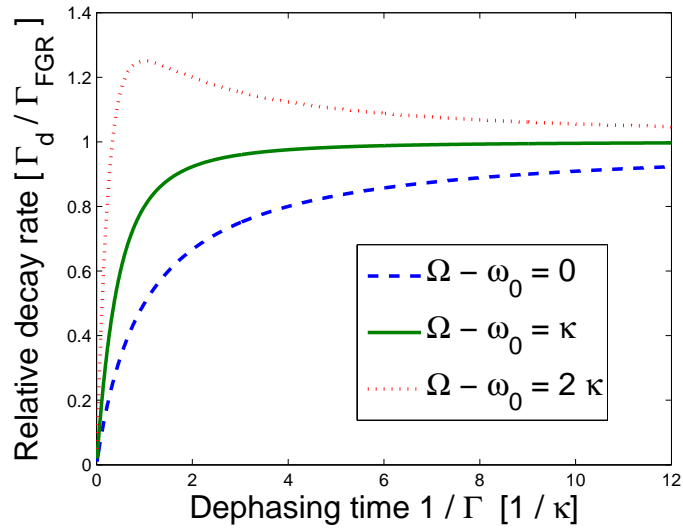Figure 2.2: The decay rate under measurements as a function of the dephasing time.

Figure 2.3: The relative decay rate under measurements, compared to Fermi's golden rule rate, as a function of the dephasing time.

So the decay rate is given by:

$$\Gamma_d(\Gamma) = \int d\omega \gamma \frac{\kappa/2\pi}{\kappa^2 + (\omega - \omega_0)^2} \frac{2\Gamma}{(\omega_k - \Omega)^2 + \Gamma^2} = \gamma \frac{\kappa + \Gamma}{(\kappa + \Gamma)^2 + (\Omega - \omega_0)^2} \quad (2.26)$$

So we get a simple analytical expression for the decay rate as a function of the dephasing rate. We notice that for $\kappa < |\Omega - \omega_0|$ we have a maximum decay rate at $\Gamma = |\Omega - \omega_0| - \kappa$. It is more common in literature to consider the measuring time as a parameter. In the system we are considering it is equal to the dephasing time ($= 1/\Gamma$). The decay rate as a function of the dephasing time is plotted in Fig. 2.2.

For a small dephasing rate, or long dephasing time, we get a constant rate which is given by Fermi's golden rule:

$$\Gamma_{FGR} = \gamma \frac{\kappa}{\kappa^2 + (\Omega - \omega_0)^2}. \quad (2.27)$$

This is the decay rate in the absence of measurements. It is interesting to look at the relative suppression, or acceleration of the decay rate under measurement compared to FGR rate. This relation is given by:

$$\frac{\Gamma_d(\Gamma)}{\Gamma_{FGR}} = \frac{\left(\kappa^2 + (\Omega - \omega_0)^2\right)(\kappa + \Gamma)}{\kappa \left((\kappa + \Gamma)^2 + (\Omega - \omega_0)^2\right)}, \quad (2.28)$$

and is plotted in Fig. 2.3.

2.4 An experimental setup: SCB coupled to two cavities/oscillators

This setup, showed in fig 2.4, is a merge of two system previously realized [18, 19]. The qubit, a Single Cooper-pair Box, is connected to two systems: one is a cavity with modes close to the qubit frequency; the other is an LC oscillator with large detuning compare to the qubit. The LC oscillator is considered as a measuring device, Quantum Capacitance Readout described in ref [19]. We will express its effect on the system just through the dephasing rate it induces on the qubit. The near resonance cavity is considered as the cause of decay/relaxation. It is connected to a transmission line which can be considered as a bath. The qubit, cavity and bath can be described by the Hamiltonian [18]:

$$H = \hbar\omega_r a^\dagger a + \frac{\hbar\Omega}{2}\sigma_z + \hbar g \left(a^\dagger \sigma_- + \sigma_+ a\right) + H_\kappa + H_\gamma.$$

Here ω_r is the frequency of the cavity mode, $\hbar\Omega$ is the qubit energy splitting, g is the coupling between the qubit and the cavity and we used the pauli matrices to describe the qubits and photon creation-annihilation operators to describe the cavity.
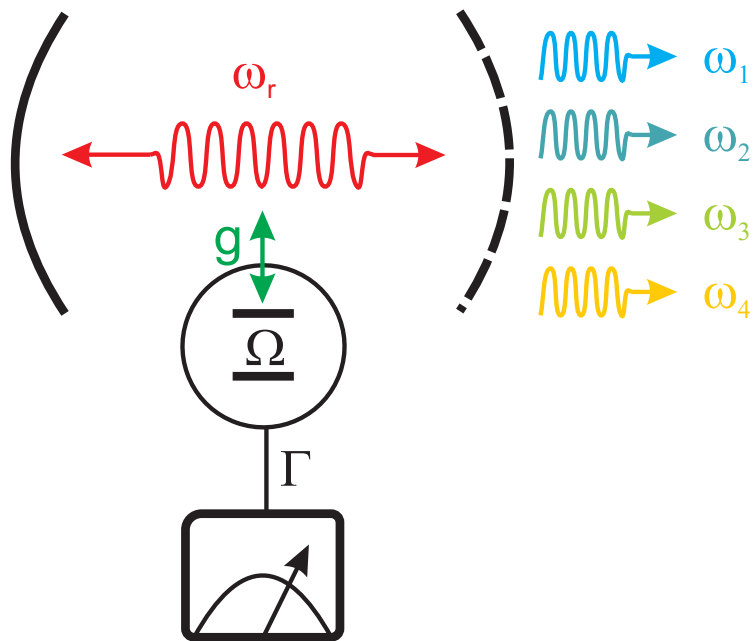
Figure 2.4: An illustration of the model for the experimental setup. A two level system (qubit) with energy splitting Ω and coupling g to a cavity with a single mode ω_r. The cavity is coupled to continuum of modes and the qubit is independently measured in a rate Γ_D.

This is using the rotating wave approximation, i.e dropping the terms $a^\dagger \sigma_+$ and $\sigma_- a$.

H_γ is the interaction of the qubits with modes outside the cavity. According to experimental results it is small and thus we neglect it.

H_κ is the interaction of the cavity mode with modes outside the cavity. It is responsible for a decay rate, κ, of the cavity mode and can be written as:

$$H_\kappa = \hbar \int d\omega \left(\sqrt{\frac{\kappa}{2\pi}} \left(a^\dagger b_\omega + b_\omega^\dagger a \right) + \omega b_\omega^\dagger b_\omega \right). \qquad (2.29)$$

Again this is written using RWA, i.e dropping terms like $a^\dagger b_\omega^\dagger$ and $b_\omega a$.

2.4.1 Diagnolize cavity-bath

To diagnolize this part we can introduce new operators [13]:

$$B_\mu = \alpha(\mu)a + \int d\omega \beta(\mu,\omega)b_\omega,$$

$$\alpha(\mu) = \frac{(\kappa/2\pi)^{1/2}}{\mu - \omega_r + i\kappa/2},$$

$$\beta(\mu,\omega) = \frac{\kappa/2\pi}{(\mu - \omega_r + i\kappa/2)(\mu - \omega + i\delta)} + \delta(\mu - \omega),$$

which obey the commutation relation:

$$\left[B_\mu, B_{\mu'}^\dagger \right] = \alpha(\mu)\alpha^*(\mu') + \int d\omega \beta(\mu,\omega)\beta^*(\mu',\omega) = \delta(\mu' - \mu).$$

Now we can calculate:

$$\begin{aligned}
\hbar \int d\mu \mu B_\mu^\dagger B_\mu &= \hbar \int d\mu \mu \left(\alpha^*(\mu)a^\dagger + \int d\omega' \beta^*(\mu,\omega')b_{\omega'}^\dagger \right) \times \\
&\qquad \left(\alpha(\mu)a + \int d\omega \beta(\mu,\omega)b_\omega \right) \\
&= \hbar \omega_r a^\dagger a + \hbar \int d\omega \left(\sqrt{\frac{\kappa}{2\pi}} \left(a^\dagger b_\omega + b_\omega^\dagger a \right) + \omega b_\omega^\dagger b_\omega \right),
\end{aligned}$$

where in the last step I used the following identities:

$$\int d\mu \mu \, |\alpha(\mu)|^2 = \omega_r,$$

$$\int d\mu \mu \alpha(\mu)\beta^*(\mu,\omega) = \sqrt{\frac{\kappa}{2\pi}},$$

$$\int d\mu \mu \beta(\mu,\omega)\beta^*(\mu,\omega') = \delta(\omega' - \omega).$$

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To insert this into the original Hamiltonian we note that:

$$a = \int d\mu\, \alpha^*(\mu) B_\mu,$$

$$b_\omega = \int d\mu\, \beta^*(\mu,\omega) B_\mu.$$

So The Hamiltonian is now:

$$H = \frac{\hbar\Omega}{2}\sigma_z + \hbar \int d\mu \left(g\alpha^*(\mu)\sigma_+ B_\mu + g\alpha(\mu) B_\mu^\dagger \sigma_- + \mu B_\mu^\dagger B_\mu \right). \quad (2.30)$$

This is equivalent to equation 2.1 with the mapping:

$$\mu \to \omega,$$
$$g^2 \left| \alpha(\mu) \right|^2 \to G(\omega),$$

and we can see from form of $\alpha(\mu)$ that the spectral density is given by

$$G(\omega) = g^2 \frac{\kappa/2\pi}{\kappa^2 + (\omega - \omega_0)^2}. \quad (2.31)$$

Equation 2.26 can be used for this system with $\gamma = g^2$.

2.4.2 Relevant Parameter Regime

Most of the parameters in this setup are controllable within a wide range. The qubit frequency Ω, the qubit - cavity coupling g, and the cavity life time $1/\kappa$ can all be set to fulfill each of the different assumptions we made [18]. But we do have some limitations. The dephasing rate, Γ, cannot made arbitrary large without effects like Landau-Zener transitions becoming dominant. On the other side we have a limit of the decay, or relaxation, rate of the qubit. We are trying to reduce (or increase in the case of AZE) the decay rate. The process which we can influence is the decay to the cavity. There is another decay process caused by the coupling of the qubit to other elements besides the cavity. We assumed that coupling of the qubit to the cavity, g, is small but it need to be bigger than other interactions. If we consider that the qubit have another decay rate of γ we can formulate our constraints as follow:

$$\gamma < \frac{g^2}{\kappa} < g < \kappa, \Gamma_{max}. \quad (2.32)$$

The second term is approximately the FGR of decay into the cavity. Since we consider g to be smaller than other relevant quantities, the next two inequalities follow.

In ref [19] the measurement time was shown to be as small as 50 ns, making $\Gamma_{max} = 0.2 GHz$. A coherence time of 500ns was measured in ref [25] making $\gamma = 20 MHz$. So it is possible to tune κ and g to fulfill the inequalities above.

2.5 Conclusion

We investigated closely a system with two sources of decoherence, one represents a zero temperature bath and the other a measurement device. The effect of the measurement device was taken as pure dephasing which agrees with the model of weak QND measurements. The bath was described using a form factor and the combined effect on the qubit was calculated, showing both QZE and AZE. Since the decay process was described by an analytic formula a more precise estimate of the jump time can be made. An experimental set up where the calculations can be checked is suggested and an analysis of realistic parameters implies that the interesting features of QZE and AZE can be seen.

Considering two decoherence processes and their joint effect is a way to shed some more light on the subject of quantum coherence and its limits. Using this formalism the borderline between the coherent quantum evolution, which is described by unitary transformation, and the non-unitary evolution which is less understood, can be investigated further. Combined with the fact that this can be experimentally examined, using techniques from quantum computations, there is great potential for characterizing further the processes of decoherence and understanding the limits of quantum mechanics. This in turn can help the development of quantum computation, especially in the context of quantum errors corrections [17] .

Chapter 3

Coupling two qubits via an Oscillator

3.1 Introduction

Entanglement is one of the most peculiar properties of quantum mechanics. The seemingly "magical" correlations between two objects which are spatially separated, is so counter-intuitive that it was used by Einstein to try disproving quantum mechanics [26]. Almost any quantum algorithm is depending in some way on entanglement [1]. It is hard to overstate its power as a QI tool. Creating entanglement is sometimes a complex problem. One needs to make the relevant objects interact with each other but also to control their interaction with the environment to avoid decoherence [27]. There are numerous ways of doing so where the main idea is to create an interaction where the state of one object affect the state of the other. There is, however, an alternative way: entanglement by measurement. In this process, the objects interact with an external object, the measurement apparatus. The interaction / measurement is done in a way that the observable, or the interaction operator, is made up from the quantum states of the two objects and have degeneracy in respect to their internal configuration.

3.2 Lagrangian

The circuit being discussed is presented in figure 3.1 Two Josephson charge qubits are capacitatively coupled to a harmonic oscillator, which is coupled to a transmission line. Through this line all measurements on the qubits are performed. We model the line as a semi-infinite line of LC-circuits in series. The working point of one Josephson junction i, can be chosen using the bias V_{gi}.

Following standard procedure [28] the capacitive energy of the circuit act as
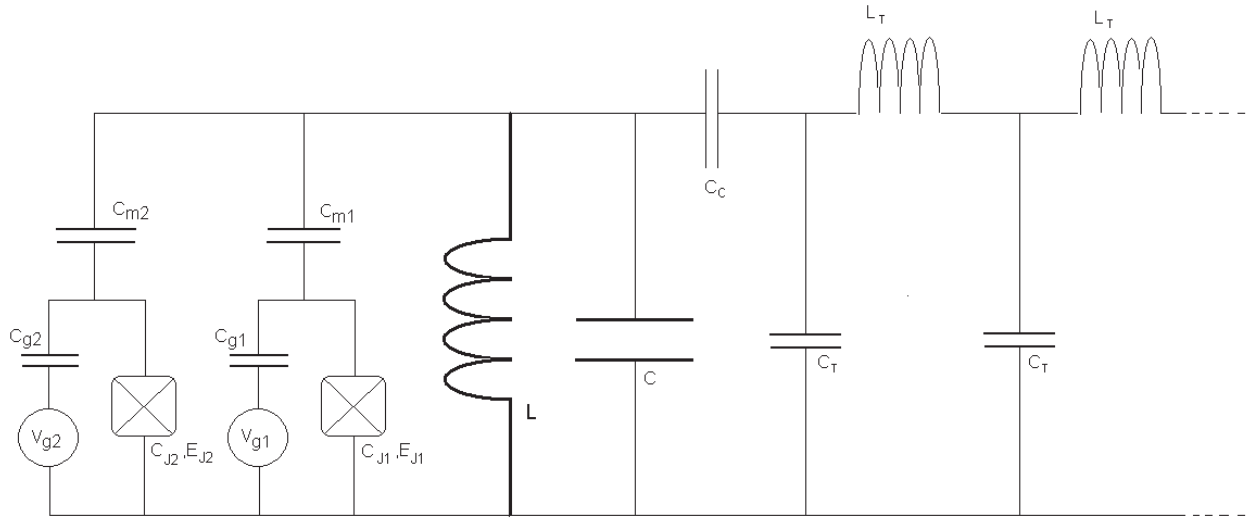
Figure 3.1: A diagram of the proposed circuit. The two SCB:s are modeled using gate capacitances, C_{g1}, C_{g2}, and Josephson Junctions having Josephson energy, E_{J1}, E_{J2} and capacitances C_{J1}, C_{J2}. The transmission line is described with as a series of LC oscillators having much lower capacitance, C_T and inductance L_T than the main LC oscillator. The coupling to the transmission line is through a small capacitance, C_c, and the coupling to the SCB:s is through another pair of small capacitances C_{m1}, C_{m2}.

the kinetic terms in the Lagrangian:

$$
\begin{aligned}
T &= \frac{C_{g1}\dot{\Phi}_{g1}^2}{2} + \frac{C_{g2}\dot{\Phi}_{g2}^2}{2}\frac{C_{J1}\dot{\Phi}_{J1}^2}{2} + \frac{C_{J2}\dot{\Phi}_{J2}^2}{2} + \frac{C_{m1}\dot{\Phi}_{m1}^2}{2} + \frac{C_{m2}\dot{\Phi}_{m2}^2}{2} \\
&+ \frac{C\dot{\Phi}_C^2}{2} + \frac{C_c\dot{\Phi}_n^2}{2} + \sum_{i=1}^{\infty}\Delta x\frac{C_T(\dot{\Phi}_i^p)^2}{2},
\end{aligned}
\tag{3.1}
$$

and the inductive part plays the part of potential energy

$$
V = \frac{\Phi_L^2}{2L} - E_{J1}\cos\left(\frac{\Phi_{J1}}{\Phi_0}\right) - E_{J2}\cos\left(\frac{\Phi_{J2}}{\Phi_0}\right) + \sum_i \Delta x\frac{(\Phi_{i+1}^p - \Phi_i^p)^2}{2L_T(\Delta x)^2}.
\tag{3.2}
$$

Applying Kirchoff's voltage law gives us the constraints

$$
\begin{aligned}
\dot{\Phi}_{g1} - \dot{\Phi}_{J1} + V_{g1} &= 0, \\
\dot{\Phi}_{g2} - \dot{\Phi}_{J2} + V_{g2} &= 0, \\
\Phi_{J1} + \Phi_{m1} - \Phi_C &= 0, \\
\Phi_{J2} + \Phi_{m2} - \Phi_C &= 0, \\
\Phi_n + \Phi_L - \Phi_1^p &= 0, \\
\Phi_C - \Phi_L &= 0,
\end{aligned}
\tag{3.3}
$$

which gives the Lagrangian for the system:

$$
\begin{aligned}
L &= \frac{C_{qb1}\dot{\Phi}_{J1}^2}{2} + \frac{C_{qb2}\dot{\Phi}_{J2}^2}{2} + \frac{(C_{osc} + C_c)\dot{\Phi}_C^2}{2} + \frac{C_c(\dot{\Phi}_1^p)^2}{2} \\
&- C_{g1}V_{g1}\dot{\Phi}_{J1} - C_{g2}V_{g2}\dot{\Phi}_{J2} - C_{m1}\dot{\Phi}_C\dot{\Phi}_{J1} - C_{m2}\dot{\Phi}_C\dot{\Phi}_{J2} - C_c\dot{\Phi}_C\dot{\Phi}_1^p \\
&- \frac{\Phi_C^2}{2L} + E_{J1}\cos\left(\frac{\Phi_{J1}}{\Phi_0}\right) + E_{J2}\cos\left(\frac{\Phi_{J2}}{\Phi_0}\right) \\
&- \sum_{i=1}^{\infty}\Delta x\left(\frac{C_T(\dot{\Phi}_i^p)^2}{2} - \frac{(\Phi_{i+1}^p - \Phi_i^p)^2}{2L_T(\Delta x)^2}\right).
\end{aligned}
\tag{3.4}
$$

The capacitances in the Lagrangian are now $C_{qbi} = C_{Ji} + C_{gi} + C_{mi}$, and $C_{osc} = C + C_{m1} + C_{m2}$. To simplify the derivation of the Hamiltonian we introduce the vector notation for our coordinates

$$
\vec{\Phi} = \begin{bmatrix} \Phi_{J1} \\ \Phi_{J2} \\ \Phi_C \\ \Phi_1^p \\ \Phi_2^p \\ \vdots \end{bmatrix},
\tag{3.5}
$$

giving us the Lagrangian on matrix form

$$
L = \frac{1}{2}\dot{\vec{\Phi}}^T C\dot{\vec{\Phi}} - C_{g1}V_{g1}\dot{\Phi}_{J1} + \frac{C_{g1}V_{g1}^2}{2} - C_{g2}V_{g2}\dot{\Phi}_{J2} + \frac{C_{g2}V_{g2}^2}{2} - \frac{\Phi_C^2}{2L}
$$

$$+ \quad E_{J1}\cos\left(\frac{\Phi_{J1}}{\Phi_0}\right) + E_{J2}\cos\left(\frac{\Phi_{J2}}{\Phi_0}\right)$$

$$- \quad \sum_{i=1}^{\infty} \Delta x \left(\frac{C_T(\dot{\Phi}_i^p)^2}{2} - \frac{(\Phi_{i+1}^p - \Phi_i^p)^2}{2L_T(\Delta x)^2}\right). \tag{3.6}$$

Here C is the mass matrix of the system

$$C = \left[\begin{array}{c|c} C_{sys} & 0 \\ \hline 0 & C_{TL} \end{array}\right], \tag{3.7}$$

where C_{sys} is the mass matrix for the circuit part of the Lagrangian

$$C_{sys} = \begin{pmatrix} C_{qb1} & 0 & -C_{m1} & 0 \\ 0 & C_{qb2} & -C_{m2} & 0 \\ -C_{m1} & -C_{m2} & C_{osc} + C_c & -C_c \\ 0 & 0 & -C_c & C_c + \Delta x C_T \end{pmatrix}.$$

Here the term $C_T\Delta x$ goes to zero as $\Delta x \to 0$, and C_{TL} represents the transmission line degrees of freedom

$$C_{TL} = \begin{bmatrix} C_T & 0 & \dots \\ 0 & C_T & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.8}$$

3.3 The Full Classical Hamiltonian

Having the Lagrangian on this form the Hamiltonian is easily obtained [28]

$$H = H_{qb} + H_{osc} + H_{TL} + H_{int}, \tag{3.9}$$

where H_{qb} contains the qubit degrees of freedom including the coupling of the qubit to the rest of the system.

The Hamiltonian is written using $D = C^{-1}$:

$$\begin{aligned} D_{sys} &= \frac{1}{(C_{osc}C_{qb2} - C_{m2}^2)C_{qb1} - C_{qb2}C_{m1}^2} \times \\ & \begin{bmatrix} C_{osc}C_{qb2} - C_{m2}^2 & C_{m1}C_{m2} & C_{m1}C_{qb2} & C_{qb2}C_{m1} \\ C_{m1}C_{m2} & C_{osc}C_{qb1} - C_{m1}^2 & C_{m2}C_{qb1} & C_{qb1}C_{m2} \\ C_{qb2}C_{m1} & C_{m2}C_{qb1} & C_{qb1}C_{qb2} & C_{qb1}C_{qb2} \\ C_{qb2}C_{m1} & C_{qb1}C_{m2} & C_{qb1}C_{qb2} & A/C_c \end{bmatrix}, \\ A &= ((C_{osc} + C_c)C_{qb2} - C_{m2}^2)C_{qb1} - C_{qb2}C_{m1}^2. \end{aligned}$$

For $C_{m2}, C_{m1} \ll C_{osc}, C_{qb1}, C_{qb2}$ this is approximatly:

$$D_{sys} = \frac{1}{C_{osc}C_{qb2}C_{qb1}} \begin{bmatrix} C_{osc}C_{qb2} & C_{m1}C_{m2} & C_{m1}C_{qb2} & C_{qb2}C_{m1} \\ C_{m1}C_{m2} & C_{osc}C_{qb1} & C_{m2}C_{qb1} & C_{qb1}C_{m2} \\ C_{qb2}C_{m1} & C_{m2}C_{qb1} & C_{qb1}C_{qb2} & C_{qb1}C_{qb2} \\ C_{qb2}C_{m1} & C_{qb1}C_{m2} & C_{qb1}C_{qb2} & A/C_c \end{bmatrix}, \tag{3.10}$$

$A = (C_{osc} + C_c)C_{qb2}C_{qb1}.$

The full Hamiltonian is given by

$$
\begin{aligned}
H_{qb} &= \frac{D_{11}}{2}(p_{J1} + C_{g1}V_{g1})^2 + \frac{D_{22}}{2}(p_{J2} + C_{g2}V_{g2})^2 \\
&+ D_{12}(p_{J1} + C_{g1}V_{g1})(p_{J2} + C_{g2}V_{g2}) \\
&+ (D_{13}p_c + D_{14}p_p)(p_{J1} + C_{g1}V_{g1}) + (D_{23}p_c + D_{24}p_p)(p_{J2} + C_{g1}V_{g2}) \\
&- E_{J1}\cos(\frac{\phi_{J1}}{\phi_0}) - E_{J2}\cos(\frac{\phi_{J2}}{\phi_0}), \\
H_{osc} &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L}, \\
H_{TL} &= \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right), \\
H_{int} &= D_{34}p_Cp_1.
\end{aligned}
$$

This gives a full classical description of the system when all the degrees of freedom are described by a canonical variable and it's conjugate momentum. The next step is to move to a quantum description which means the variables describing degrees of freedom and their conjugate momentum become operators with a commutation relation $[\phi_k, p_l] = i\hbar\delta_{kl}$, when the indices k and l take the values: J1, J2, C, p1 and so on.

3.4 The Hamiltonian of the Two Qubits

The Hamiltonian in the last section, written using operators, gives a quantum description of the system but it is not easy to work with. We want to write it in a better form. The first step is using Pauli matrices for the qubit subspace.

3.4.1 Projection On Charge Basis

The way to introduce Pauli matrices is to first project the subspace of the two qubits to the charge basis, using the excess charge as our quantum number. Assuming that the excess charge can be either 0 or 1 Cooper pair (= 2e), we get two levels/states for each qubit, or four altogether.

We can first look at the terms depending on the qubit phase, like $E_{Jk}\cos\left(\frac{\Phi_{Jk}}{\Phi_0}\right)$. From the commutation relation of ϕ and p it follows that:

$$
\left[p_k, \exp\left(\pm i\frac{\Phi_{Jk}}{\Phi_0}\right)\right] = \pm i\hbar\left(\frac{i}{\Phi_0}\right)\exp\left(\pm i\frac{\Phi_{Jk}}{\Phi_0}\right) = \mp 2e\exp\left(\pm i\frac{\Phi_{Jk}}{\Phi_0}\right).
$$

So if we look at the action of $\exp\left(\pm i\frac{\Phi_{Jk}}{\Phi_0}\right)$ on the eigenstates, $|n_k\rangle$ of p_k, where n is the number of Cooper pairs, we find that it is a shift operator:

$$\exp\left(\pm i\frac{\Phi_{Jk}}{\Phi_0}\right)|n_k\rangle = |n_k \mp 1\rangle\,,$$

$$
\begin{aligned}
E_{Jk}\cos\left(\frac{\Phi_{Jk}}{\Phi_0}\right)|n_k,n_l\rangle &= E_{Jk}\frac{\exp\left(i\frac{\Phi_{Jk}}{\Phi_0}\right)+\exp\left(-i\frac{\Phi_{Jk}}{\Phi_0}\right)}{2}|n_k,n_l\rangle \\
&= E_j\frac{E_{Jk}}{2}\left(|n_k+1,n_l\rangle+|n_k-1,n_l\rangle\right).
\end{aligned}
$$

Since operators acting on different qubits commute, the only quantum number affected is the one related to the operator. In the relevant subspace, this can be written as tensor product of pauli matrices:

$$
\begin{aligned}
E_{J1}\cos\left(\frac{\Phi_{J1}}{\Phi_0}\right) &\rightarrow \frac{E_{J1}}{2}\left[\sigma_x^{(1)}\otimes I^{(2)}\right],\\
E_{J2}\cos\left(\frac{\Phi_{J2}}{\Phi_0}\right) &\rightarrow \frac{E_{J2}}{2}\left[I^{(1)}\otimes\sigma_x^{(2)}\right].
\end{aligned}
$$

Now let's look on the terms involving the qubit charge.
We use the following definitions/notation:

$$
\begin{aligned}
E_{Ci} &= \frac{e^2 D_{ii}}{2} =\simeq \frac{e^2}{2C_{qbi}},\\
\kappa_i &= D_{i3}/D_{ii} \simeq \frac{C_{mi}}{C_{osc}},\\
n_i &= \frac{p_{Ji}}{e},\\
n_p &= \frac{p_p+p_c}{e},\\
m_i' &= \frac{C_{gi}V_{gi}}{e},\\
T &= D_{12}e^2 \simeq \frac{e^2 C_{m1}C_{m2}}{C_{qb1}C_{qb2}C_{osc}}.
\end{aligned}
$$

The qubit charge terms can be now rewritten

$$
\begin{aligned}
H_{qb} &= E_{C1}(n_1+m_1')^2 + E_{C2}(n_2+m_2')^2 + T(n_1+m_1')(n_2+m_2')\\
&\quad + 2E_{C1}\kappa_1 n_p(n_1+m_1') + 2E_{C2}\kappa_2 n_p(n_2+m_2')\\
&= n_1^2 E_{C1} + n_2^2 E_{C2} + n_1(2E_{C1}(m_1'+\kappa_1 n_p)+Tm_2')\\
&\quad + n_2(2E_{C2}(m_2'+\kappa_2 n_p)+Tm_1') + Tn_1 n_2\\
&\quad + E_{C1}(m_1')^2 + E_{C2}(m_2')^2 + Tm_1'm_2' + 2E_{C1}\kappa_1 n_p m_1' + 2E_{C2}\kappa_2 n_p m_2'.
\end{aligned}
$$

Since we projected on the charge basis, the charge operators becomes numbers: 0 or -2. We calculate the part of the Hamiltonian depending on the charge for the 4 cases. The terms not containing n_1 or n_2 will appear in all cases. This can be seen as an additional unity matrix and can be ignored (this is equivalent to setting the energy zero point). For the case $n_1 = n_2 = 0$ this is the only term so we get 0. The other 3 cases are:

$$
\begin{aligned}
(n_1 = -2; n2 = 0) &\rightarrow -4E_{C1}\left(m_1' - 1 + \kappa_1 n_p\right) - 2Tm_2' = A, \\
(n_1 = 0; n2 = -2) &\rightarrow -4E_{C2}\left(m_2' - 1 + \kappa_2 n_p\right) - 2Tm_1' = B, \\
(n_1 = -2; n2 = -2) &\rightarrow -4E_{C1}\left(m_1' - 1 + \kappa_1 n_p\right) - 4E_{C2}\left(m_2' - 1 + \kappa_2 n_p\right) \\
&\quad -2Tm_2' - 2Tm_1' + 4T = A + B + 4T = C.
\end{aligned}
$$

To put this in Pauli matrix representation: $-\frac{E_{el1}}{2}\left(\sigma_z^{(1)} \otimes I^{(2)}\right) - \frac{E_{el2}}{2}\left(I^{(1)} \otimes \sigma_z^{(2)}\right) + g\left(\sigma_z^{(1)} \otimes \sigma_z^{(2)}\right) + I\left(I^{(1)} \otimes I^{(2)}\right)$, where the superscript on operators note on which qubit it's acting.

We need to solve 4 equations with 4 variables:

$$
\begin{aligned}
\frac{E_{el1}}{2} + \frac{E_{el2}}{2} + g + I &= 0, \\
\frac{E_{el1}}{2} - \frac{E_{el2}}{2} - g + I &= A, \\
-\frac{E_{el1}}{2} + \frac{E_{el2}}{2} - g + I &= B, \\
-\frac{E_{el1}}{2} - \frac{E_{el2}}{2} + g + I &= C.
\end{aligned}
$$

Using $m_i = m_i' - 1$, the solution is:

$$
\begin{aligned}
E_{el1} &= 4E_{C1}\left(m_1 + \kappa_1 n_p\right) + 2Tm_2, \\
E_{el2} &= 4E_{C2}\left(m_2 + \kappa_2 n_p\right) + 2Tm_1, \\
g &= T, \\
I &= -2E_{C1}\left(m_1 + \kappa_1 n_p\right) - Tm_2 - 2E_{C2}\left(m_2 + \kappa_2 n_p\right) - Tm_1 - T \\
&= -E_{el1}/2 - E_{el2}/2 - T.
\end{aligned}
$$

So we can write the qubit Hamiltonian in the form

$$
\begin{aligned}
H_{qb} &= -\frac{E_{el1}}{2}\sigma_z^1 - \frac{E_{J1}}{2}\sigma_x^1 - \frac{E_{el2}}{2}\sigma_z^2 - \frac{E_{J2}}{2}\sigma_x^2 + T\sigma_z^1\sigma_z^2 + E_0 I, \\
E_0 &= Tm_1 m_2 + E_{C1}(m_1^2 + 1 + 2\kappa_1 n_p m_1) + E_{C2}(m_2^2 + 1 + 2\kappa_2 n_p m_2).
\end{aligned}
$$

The superscript on operators denotes on which qubit it's acting on, and there's an implicit tensor product with unity for the single operators.

3.4.2 Neglecting the Direct Interaction Term

Now we can notice that the qubit Hamiltonian, before considering the measurement dynamics, is composed of two single qubit Hamiltonians and a direct qubit-qubit interaction term. This term is not coupled to the measurement device. Since the purpose of our system is to create entanglement by measurement which is controllable, this term is an interruption. We can look at its magnitude relative to the other parts of the Hamiltonian:

$$
\begin{aligned}
\frac{T}{\Delta E_i} &= \frac{D_{12}e^2}{\sqrt{E_{eli}^2 + E_{Ji}^2}} < \frac{D_{12}e^2}{E_{Ji}} = \frac{e^2 C_{m1} C_{m2}}{((C_{osc} C_{qb2} - C_{m2}^2) C_{qb1} - C_{qb2} C_{m1}^2) E_{Ji}} \\
&\simeq \frac{e^2 C_{m1} C_{m2}}{C_{osc} C_{qb2} C_{qb1} E_{Ji}} = \frac{4 E_{C1} E_{C2} C_{m1} C_{m2}}{e^2 C_{osc} E_{Ji}} \ll 1.
\end{aligned}
$$

Putting actual numbers used in an experimental set up, we can make this figure small enough to drop the whole term. A detailed analysis of this approximation including the use of perturbation theory is given in appendix A.

3.4.3 Diagonalization

Diagonalizing the qubit Hamiltonian means that we change basis for the Hilbert subspace of the two qubits by performing a unitary transformation. One can do this, in general, without affecting the rest of the Hilbert space. In our case the transformation depends on the state of the oscillator and thus making the transformation operator a tensor product acting on both Hilbert spaces. Since the oscillator has much lower frequency, the dynamics are on different time scales and this method holds.

The diagonal Hamiltonian can be given by the formula $H_0 = DHD^{-1}$. Where D is made from the eigenstates of H. The easiest way of representing the single qubit eigenstates is to use a mixing angle:

$$
\begin{aligned}
\theta_i &= \Theta(E_{eli}) \left(\arctan\left(\frac{E_{Ji}}{E_{eli}}\right) \right) + \Theta(-E_{eli}) \left(\pi + \arctan\left(\frac{E_{Ji}}{E_{eli}}\right) \right), \\
\Delta E_i &= \sqrt{E_{eli}^2 + E_{Ji}^2}, \\
|0\rangle &= \begin{pmatrix} \cos\left(\frac{\theta_i}{2}\right) \\ \sin\left(\frac{\theta_i}{2}\right) \end{pmatrix}, \\
|1\rangle &= \begin{pmatrix} -\sin\left(\frac{\theta_i}{2}\right) \\ \cos\left(\frac{\theta_i}{2}\right) \end{pmatrix}.
\end{aligned}
$$

The rather complicated definition of θ is due to the fact that we are interested in the region where E_{el} is close to zero and it's sign is changing. This makes the argument of arctan go between infinity and minus infinity. The above definition is continuous and has a continuous derivative in that region.

Another way to calculate D is to look at it as a rotation, of each qubit Hilbert space, around the Y axis in an angle θ_i.

So we have:

$$D = R_y^{(1)}(\theta_1) \otimes R_y^{(2)}(\theta_2) =$$
$$\begin{pmatrix} \cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) & -\cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) & -\cos\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & \sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) \\ \cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) & \cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) & -\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & -\cos\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) \\ \cos\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & -\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & \cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) & -\cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) \\ \sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & \cos\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right) & \cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) & \cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) \end{pmatrix}.$$

The diagonal Hamiltonian can be written as:

$$H_0 = DHD^{-1} = \Delta E_1 \sigma_z^{(1)} + \Delta E_2 \sigma_z^{(2)}.$$

The 4 eigenstates can be written as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and the energies are $-\Delta E_1 - \Delta E_2, \Delta E_1 - \Delta E_2, \Delta E_2 - \Delta E_1$ and $\Delta E_1 + \Delta E_2$.

3.5 The qubit - oscillator Hamiltonian

So the Hamiltonian of the full system can now be written as

$$\begin{aligned} H &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) + D_{34}p_C p_1 \\ &\quad + \frac{1}{2}\Delta E_1(n_p)\sigma_z^{(1)} + \frac{1}{2}\Delta E_2(n_p)\sigma_z^{(2)}. \end{aligned} \qquad (3.11)$$

We can expand the qubits energies, $\Delta E_i(n_p)$ around the degeneracy point $n_p = 0$:

$$\Delta E_i = E_{Ji} + \frac{8E_{ci}^2\kappa_i^2}{E_{Ji}}(p_C + p_1)^2$$

So the Hamiltonian can be written

$$\begin{aligned} H &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) + D_{34}p_C p_1 \\ &\quad + E_{J1}\sigma_z^{(1)} + E_{J2}\sigma_z^{(2)} \\ &\quad + 4\left(\frac{E_{c1}^2\kappa_1^2}{E_{J1}}\sigma_z^{(1)} + \frac{E_{c2}^2\kappa_2^2}{E_{J2}}\sigma_z^{(2)}\right)(p_C + p_1)^2. \end{aligned} \qquad (3.12)$$

We consider the following approximations; weak coupling : $C_{m1}, C_{m2}, C_c \ll C_{osc}, C_{qb1}, C_{qb2}$.

Close to resonance: $p_1 \ll p_C$ which also implies $T \ll E_{Ji}$.

So we can write it down as:

$$
\begin{aligned}
H &= \left(\frac{1}{2C_{osc}} + 4 \left(\frac{E_{c1}^2 \kappa_1^2}{E_{J1}} \sigma_z^{(1)} + \frac{E_{c2}^2 \kappa_2^2}{E_{J2}} \sigma_z^{(2)} \right) \right) p_C^2 + \frac{\Phi^2}{2L} \qquad (3.13) \\
&+ \frac{1}{2C_c} p_1^2 + \frac{1}{\Delta x} \sum_i \left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\Phi_{i+1}^p - \Phi_i^p)^2}{2L_T} \right) + \frac{1}{C_{osc}} p_C p_1 .
\end{aligned}
$$

The purpose of this setup is to produce a symmetry in the measurement of two qubits. From the expressions above we can see that this can be done quite easily. The expression determining the coupling between the qubits and the oscillator is $\frac{E_{ci}^2 \kappa_i^2}{E_{Ji}} \simeq \frac{e^4 C_{mi}^2}{4 E_{Ji} C_{qbi}^2 C_{osc2}}$. We can make it equal for the two qubits by tuning E_{Ji} to fulfill the equation $\frac{E_{c1}^2 \kappa_1^2}{E_{J1}} = \frac{E_{c2}^2 \kappa_2^2}{E_{J2}}$. By that symmetry from the "point of view" of the oscillator will be created.

3.6 Equations of Motion

The majority of this section is taken from the calculations done in ref [28]. When performing readout we are interested in the way our circuit reflects the incoming modes of the transmission line. Hence we derive the coupled equations of motion for the circuit and transmission line operators. This in turn gives us the scattering matrix for the circuit.

3.6.1 Transmission line

We start by considering the Heisenberg equations of motion for the transmission line degrees of freedom which are not coupled to the circuit. For $i \geq 2$ we have

$$
\dot{p}_i = \left[p_i, \frac{\Delta x}{2 L_T} \frac{(\Phi_{i+1} - \Phi_i)^2 - (\Phi_i - \Phi_{i-1})^2}{(\Delta x)^2} \right] \rightarrow \frac{\mathrm{d}x}{L_T} \frac{\partial^2 \Phi_i}{\partial x^2} \qquad (3.14)
$$

where

$$
p_i = C_T \mathrm{d}x \dot{\Phi}_i . \qquad (3.15)
$$

The phase in the transmission lines thus obeys the massless scalar Klein-Gordon equation

$$
\frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{L_T C_T} \frac{\partial^2 \Phi}{\partial x^2} = 0 \qquad (3.16)
$$

which has the formal solution

$$
\Phi = \Phi^{in} \left(\frac{x}{v} + t \right) + \Phi^{out} \left(-\frac{x}{v} + t \right) , \qquad (3.17)
$$

where $v = 1/\sqrt{L_T C_T}$. By differentiating equation (3.17) we can get a relation between the partial derivatives

$$
-\frac{1}{L_T} \frac{\partial \Phi}{\partial x} = \sqrt{\frac{C_T}{L_T}} \left(\frac{\partial \Phi}{\partial t} - 2 \frac{\partial \Phi^{in}}{\partial t} \right) . \qquad (3.18)
$$

The equations above, describing the transmission line dynamics, are the well known electrodynamics equations for an electromagnetic field in one dimension. They were analyzed and used in many contexts, using a few formalism to describe the state of the field. The formalism I will want to use eventually is the coherent Glauber state, which is base on so called second quantization using creation annihilation operators. Using the Glauber state formalism we can made contact with the three quantities required: the frequency, the phase and the photon number/intensity.

3.6.2 Circuit (Oscillator + Transmission line)

For this part we use the part of the Hamiltonian containing Oscillator and Transmission operator dropping the perturbation terms and higher order corrections. This is given by:

$$
\begin{aligned}
H &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) \\
&+ D_{34}p_C p_1 + X(p_C + p_1)^2
\end{aligned}
$$

where $X = 4\left(\frac{E_{c1}^2\kappa_1^2}{E_{J1}}\sigma_z^{(1)} + \frac{E_{c2}^2\kappa_2^2}{E_{J2}}\sigma_z^{(2)}\right)$ is a coefficient depending on the qubits state.

The Heisenberg equations of motion then:

$$
\dot{p_C} = -\frac{i}{\hbar}\left[p, \frac{1}{2L}\Phi_C^2\right] = -\frac{1}{L}\Phi_C, \tag{3.19}
$$

$$
\begin{aligned}
\dot{p_1} &= -\frac{i}{\hbar}\left[p_1, \frac{1}{2L_T}\frac{(\Phi_2^{pump} - \Phi_1^{pump})^2}{\Delta x}\right] \rightarrow \frac{1}{L_T}\frac{\partial\Phi_p}{\partial x} \\
&= -\sqrt{\frac{C_T}{L_T}}\left(\frac{\partial\Phi_1}{\partial t} - 2\frac{\partial\Phi^{in}(x=0)}{\partial t}\right), \tag{3.20}
\end{aligned}
$$

$$
\dot{\Phi}_C = -\frac{i}{\hbar}\left[\Phi_C, \frac{D_{33}}{2}p_C^2 + D_{34}p_C p_1 + X(p_C + p_1)^2\right] = (D_{34} + 2X)(p_C + p_1),
$$

$$
\begin{aligned}
\dot{\Phi}_1^p &= -\frac{i}{\hbar}\left[\Phi_1, \frac{D_{44}}{2}p_1^2 + D_{34}p_C p_1 + X(p_C + p_1)^2\right] \\
&= (D_{44} + 2X)p_1 + (D_{34} + 2X)p_C. \tag{3.21}
\end{aligned}
$$

Using $D_{33} = D_{34}$.

Due to the linearity of these equations it is easier solve them in Fourier space, where the set of four coupled equations are easily reduced to two, only containing the phase operators of the oscillator and the transmission line:

$$
i\omega\Phi_C(\omega) = (D_{34} + 2X)\left(\frac{i}{L\omega}\Phi_C(\omega) - \sqrt{\frac{C_T}{L_T}}\left(\Phi_1(\omega) - 2\Phi^{in}(\omega)\right)\right), \tag{3.22}
$$

$$
i\omega\Phi_1(\omega) = -(D_{44} + 2X)\sqrt{\frac{C_T}{L_T}}\left(\Phi_1(\omega) - 2\Phi^{in}(\omega)\right) + (D_{34} + 2X)\frac{i}{L\omega}\Phi_C(\omega).
$$

This two equations allow us to solve them in Fourier space

$$
\begin{aligned}
N^{g/e}(\omega) &= (D_{34} + 2X)^2 + ((D_{44} + 2X) - iZ_0\omega) \times \\
&\quad \left(L\omega^2 - (D_{34} + 2X)\right), \\
p_C^{g/e}(\omega) + p_1^{g/e}(\omega) &= \frac{-2iL\omega^3}{N^{g/e}(\omega)}\Phi_1^{in}(\omega), \\
\Phi_1^{out}(\omega) &= \frac{N^{g/e}(-\omega)}{N^{g/e}(\omega)}\Phi_1^{in}(\omega) = S(\omega)\Phi_1^{in}(\omega),
\end{aligned}
\tag{3.23}
$$

where $Z_0 = \sqrt{L_T/C_T}$ is the characteristic impedance of the transmission line. Since there is no dissipation in the lumped circuit we have $|\Phi^{out}(\omega)| = |\Phi^{in}(\omega)|$. In the weak coupling regime $C_c \ll C_{osc}$ we have $\arg(S(\omega)) = 0$ except close to the resonance $\omega \approx 1/\sqrt{LC_\Sigma}$.

3.7 Measurement

In the quantum capacitance setup, a measurement is done on the phase of the reflected signal. As we saw, this phase is dependent on the states of the qubits via the observable $\hat{X} = 4\left(\frac{E_{c1}^2\kappa_1^2}{E_{J1}}\sigma_z^{(1)} + \frac{E_{c2}^2\kappa_2^2}{E_{J2}}\sigma_z^{(2)}\right)$.

Since the E_J is tunable the coefficients can be set to be equal or different. When they are equal the measurement cannot distinguish the states $|01\rangle$ and $|10\rangle$.

To put this in more exact language we need to look more closely on the measurement process and its back action. The Hamiltonian of the combined system contains a term coupling the quantum operator of the LC oscillator with the qubits operator above. As shown in [19] this coupling is a source of a change in the reflected phase and also for dephasing of the qubit. The measurement time, which is the inverse of the dephasing rate, is dependent on the coupling strength. We want to look into the effect on the qubit.

The combined system can be described by the combined density matrix ρ. If one wants to study only part of the system one might use the reduced density matrix, which is given by tracing all other degrees of freedom from the combined density matrix: $\rho_s = Tr_s(\rho)$. We are interested in the state of the two qubits so the reduced density matrix can be written as a 4 by 4 matrix using the bases $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

The reduced density matrix obeys the Lindblad Master equation:

$$
\dot{\rho}_S = \frac{1}{i\hbar}[H_S, \rho_S] + \sum_k \left[L_k\rho_S L_k^\dagger - \frac{1}{2}\{\rho_S, L_k^\dagger L_k\}\right],
\tag{3.24}
$$

where H_S is the Hamiltonian of the system and L_k are the Lindblad operators which, in our case, are given by $\hat{X}$. One can see that any parts of ρ_s which commutes with $\hat{X}$ will not be affected by this dynamics. Other elements will decay exponentially. A more detailed analysis of such dynamics is given in chapter 2.

In order to see how this applies to our case we define:

$$g = \frac{E_{c1}^2 \kappa_1^2}{E_{J1}} + \frac{E_{c2}^2 \kappa_2^2}{E_{J2}},$$

$$\gamma = \frac{E_{c1}^2 \kappa_1^2}{E_{J1}} - \frac{E_{c2}^2 \kappa_2^2}{E_{J2}},$$

$$\hat{B} = \sigma_z^{(1)} + \sigma_z^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$\hat{C} = \sigma_z^{(1)} - \sigma_z^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$(3.25)$$

We get

$$\hat{X} = 2 \left(g\hat{B} + \gamma\hat{C} \right). \qquad (3.26)$$

The term containing $g\hat{B}$ will cause the six "most off diagonal" components to vanish:

$$\begin{pmatrix} & x & x & x \\ x & & & x \\ x & & & x \\ x & x & x & \end{pmatrix}. \qquad (3.27)$$

Which is what we want for entanglement.

Those components will die exponentially with a rate $\Gamma_g \sim g^2$ [19], so the time we need to induce this process (the measurement time) would be $T_g \sim \frac{1}{g^2}$.

But the term containing $\gamma\hat{C}$ will cause also another 2 elements to vanish:

$$\begin{pmatrix} & & x & \\ & x & & \\ & & & \\ & & & \end{pmatrix}. \qquad (3.28)$$

Those components will die exponentially with a rate $\Gamma_\gamma \sim \gamma^2$, so the factor caused be dephasing after time T_g will be: $e^{-T_g \Gamma_\gamma} = e^{-\frac{\gamma^2}{g^2}}$. For $\gamma \ll g$ this is around unity.

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3.8 Entanglement procedure

To entangle the qubits we follow these steps:

1. Start with both qubits in the ground state:

$$\Psi = |0\rangle\,|0\rangle \quad ; \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.29}$$

2. We do a Hadamard rotation on each qubit

$$\Psi = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) = \frac{1}{2}(|0\rangle\,|0\rangle + |0\rangle\,|1\rangle + |1\rangle\,|0\rangle + |1\rangle\,|1\rangle)$$

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.30}$$

3. We make a measurement with similar coefficients as discussed above. A simplified description in terms of the state vector is to say that in a probability $1/2$ we'll get the result corresponding to $|0\rangle\,|1\rangle + |1\rangle\,|0\rangle$ which is a maximally entangled state. For other results we start over from step 1.

To describe this in terms of the density matrix we'll say that the dephasing will give the following block diagonal matrix:

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.31}$$

The readout of the measurement will let us know in which block we are. For unwanted result we start over. Assuming the wanted result, we get the state:

$$\Psi = \frac{1}{\sqrt{2}}(|0\rangle\,|1\rangle + |1\rangle\,|0\rangle) \quad ; \quad \rho = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

This is one of the Bell states which are maximally entangled states.

In case we want to use this state to violate Bell's inequalities, we perform 3 more steps:

4. We make a rotation on each qubit separately, using the microwave pulses. This is equivalent to rotating the detector axis [27].

5. We change the magnetic flux so E_j will change in a way that the coefficients of the measurement operator will be different.

6. We make a final measurement.

3.9 Conclusion

Two SCB qubits connected capacitively to an LC oscillator were analyzed in a fully quantum mechanical way. The Hamiltonian of the whole system is derived and the evolution of the system is determined by using Heisenberg equations of motions. The result shows that one can perform measurement on the sum of the states and by that create entanglement. This can be used by a simple procedure to get a Bell state.

This set up has a few advantages as an entanglement creating system. The first is the fact that the interaction responsible for entanglement can be switched on and off. This means that it can be used, for example, for quantum gates, but also when one exploit the Zeno effect it can keep the qubits in the entangled subspace. This can be an implementation of the Quantum Zeno Error Correction [17]. Another advantage is that the measurement device is built-in. This fact enables us to perform individual readout with no change of hardware.

The modularity of SCB qubits connected capacitively suggest that more elaborated set ups might be possible. One might connect a few qubits to the same oscillator in order to create a register where each qubit can be read independently. Connecting each pair in a group of qubits to different oscillator can be used to make intricate gates in a circuit. The main principles in calculating the dynamics of such system are shown and it is basically a matter of applying them to the specific system.

Chapter 4

Conclusion

In order to build a quantum computer one need some qubits, the ability to connect and measure them and that those qubits will remain coherent for long enough time. All of these ingredients were discussed in this thesis. Naturally there are no simple solutions for any of the problems in realizing a quantum computer. What can be done is to devise systems in which such solutions can be tested, and to investigate the theoretical questions standing in the heart of these solutions. Of those questions the one which, in my opinion, is the most intriguing and have a major practical significance, is whether the decoherence process, and its side effect -quantum errors, can be efficiently stopped.

The investigation of the Zeno effect gives us a unique insight of the issues concerning decoherence. Even though the term decoherence might have come to use many years after the discovery of the Zeno effect, it is the most profound attempt to explain the so called collapse of the wave function. This "collapse" is in a way the origin of the Zeno effect. The decay process, which is another important part of the original Zeno effect, is another example of decoherence process.

There is much more to do in this field, both experimentally and theoretically. A lot of mysteries need to be explained, many technical obstacles need to be overcome. These have to come hand in hand; there is very little likelihood of a major progress in one without the support of the other. The incentive is double: understanding some of the basic laws of nature and creating devices with high practical value.

Appendix A

Dealing with the direct interaction term

The qubits Hamiltonian contained a direct qubit-qubit coupling in the form of $T\sigma_z^{(1)\sigma_z^{(2)}}$ which we dropped. Since we later did a change of basis this term should be written as:

$$
\begin{aligned}
H_I &= T \begin{pmatrix}
\cos\theta_2\cos\theta_1 & -\cos\theta_1\sin\theta_2 & -\cos\theta_2\sin\theta_1 & \sin\theta_2\sin\theta_1 \\
-\cos\theta_1\sin\theta_2 & -\cos\theta_2\cos\theta_1 & \sin\theta_2\sin\theta_1 & \cos\theta_2\sin\theta_1 \\
-\cos\theta_2\sin\theta_1 & \sin\theta_2\sin\theta_1 & -\cos\theta_2\cos\theta_1 & \cos\theta_1\sin\theta_2 \\
\sin\theta_2\sin\theta_1 & \cos\theta_2\sin\theta_1 & \cos\theta_1\sin\theta_2 & \cos\theta_2\cos\theta_1
\end{pmatrix} \\
&= T\left(\cos\theta_1\cos\theta_2\sigma_z^{(1)}\sigma_z^{(2)} - \cos\theta_1\sin\theta_2\sigma_z^{(1)}\sigma_x^{(2)}\right. \\
&\qquad \left. -\sin\theta_1\cos\theta_2\sigma_x^{(1)}\sigma_z^{(2)} + \sin\theta_1\sin\theta_2\sigma_x^{(1)}\sigma_x^{(2)}\right) \\
&= T\left(\cos\theta_1\sigma_z^{(1)} - \sin\theta_1\sigma_x^{(1)}\right)\left(\cos\theta_2\sigma_z^{(2)} - \sin\theta_2\sigma_x^{(2)}\right).
\end{aligned}
$$

We can deal with this term in two ways, including it in the Hamiltonian of the full system as it is and look at its impact; or treating it a perturbation and change the basis a bit.

A.1 First order perturbation

As shown above the perturbation parameter is small, so a first order perturbation might be enough.

Note that we have also $H_0 \sim \Delta E_1 - \Delta E_2$. In case the qubits are similar, the difference between the energies can vanish and we need to resort to degenerate perturbation theory. We do not deal with this case here.

The perturbation element is given above. The diagonal elements are the first order energy corrections.

The first order corrections to the states are given by:

$$|n\rangle = |n^{(0)}\rangle + \sum_{k \neq n} |k^{(0)}\rangle \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}.$$

So in our case

$$|0\rangle = |00\rangle + T\left(|01\rangle\frac{cos(\theta_1)sin(\theta_2)}{\Delta E_2} + |10\rangle\frac{sin(\theta_1)cos(\theta_2)}{\Delta E_1} - |11\rangle\frac{sin(\theta_1)sin(\theta_2)}{\Delta E_1 + \Delta E_2}\right),$$

$$|1\rangle = |01\rangle + T\left(-|00\rangle\frac{cos(\theta_1)sin(\theta_2)}{\Delta E_2} + |10\rangle\frac{sin(\theta_1)sin(\theta_2)}{\Delta E_2 - \Delta E_1} + |11\rangle\frac{sin(\theta_1)cos(\theta_2)}{\Delta E_1}\right),$$

$$|2\rangle = |10\rangle + T\left(-|00\rangle\frac{sin(\theta_1)cos(\theta_2)}{\Delta E_1} + |01\rangle\frac{sin(\theta_1)sin(\theta_2)}{\Delta E_1 - \Delta E_2} + |11\rangle\frac{cos(\theta_1)sin(\theta_2)}{\Delta E_2}\right),$$

$$|3\rangle = |11\rangle + T\left(+|00\rangle\frac{sin(\theta_1)sin(\theta_2))}{\Delta E_1 + \Delta E_2} - |01\rangle\frac{sin(\theta_1)cos(\theta_2)}{\Delta E_1} - |10\rangle\frac{cos(\theta_1)sin(\theta_2)}{\Delta E_2}\right).$$

So the new basis can be created by the matrix:

$$I + T\left(-\frac{\cos\theta_1\sin\theta_2}{\Delta E_2}i\sigma_y^{(2)} - \frac{sin(\theta_1)cos(\theta_2)}{\Delta E_1}i\sigma_y^{(1)}\right.$$
$$\left. + \frac{sin(\theta_1)sin(\theta_2)}{\Delta E_1^2 - \Delta E_2^2}\left(\Delta E_2 i\sigma_x^{(1)}\sigma_y^{(2)} - \Delta E_1 i\sigma_y^{(1)}\sigma_x^{(2)}\right)\right).$$

In this base the qubit hamiltonian is diagonal up to terms of $O\left(\frac{T}{\Delta E_i}^2\right)$. We can use this base to analyze the full Hamiltonian.

In the first order perturbed base the qubit hamiltonain is diagonal up to terms of $O\left(\frac{T}{\Delta E_i}^2\right)$. We can expand the energies as we did before, now adding the first order correction. Since at the degeneracy point the first order correction and its first derivative vanish, we get the second order term: $\frac{32E_{c1}E_{c2}\kappa_1\kappa_2}{E_{j1}E_{j2}}$

So in this base the Hamiltonian of the full system is

$$H = \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) + D_{34}p_Cp_1$$

$$+E_{j1}\sigma_z^{(1)} + E_{j2}\sigma_z^{(2)}$$

$$-\left(\frac{4E_{c1}^2\kappa_1^2}{E_{j1}}\sigma_z^{(1)} + \frac{4E_{c2}^2\kappa_2^2}{E_{j2}}\sigma_z^{(2)} + T\frac{16E_{c1}E_{c2}\kappa_1\kappa_2}{E_{j1}E_{j2}}\sigma_z^{(1)}\sigma_z^{(2)}\right)(p_C + p_1)^2$$

$$+O\left(\frac{T}{\Delta E_i}^2\right).$$

A.2 Without Perturbation Corrections

Putting the interaction term as it is we get

$$
\begin{aligned}
H &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) \\
&+ D_{34}p_C p_1 + \frac{1}{2}\Delta E_1(n_p)\sigma_z^{(1)} + \frac{1}{2}\Delta E_2(n_p)\sigma_z^{(2)} \\
&+ T\left(\cos\theta_1\sigma_z^{(1)} - \sin\theta_1\sigma_x^{(1)}\right)\left(\cos\theta_2\sigma_z^{(2)} - \sin\theta_2\sigma_x^{(2)}\right).
\end{aligned}
$$

Expanding the interaction term around the degeneracy point, as we did for the qubits energies above:

$$
\begin{aligned}
H_I &= T\left(\sigma_x^{(1)}\sigma_x^{(2)} + \left(\frac{4E_{C1}\kappa_1}{E_{j1}}\sigma_z^{(1)}\sigma_x^{(2)} + \frac{4E_{C2}\kappa_2}{E_{j2}}\sigma_z^{(2)}\sigma_x^{(1)}\right)n_p \right. \\
&+ \left. \left(\frac{16E_{c1}E_{c2}\kappa_1\kappa_2}{E_{j1}E_{j2}}\sigma_z^{(1)}\sigma_z^{(2)} - \left(\frac{8E_{c1}^2\kappa_1^2}{E_{j1}^2} + \frac{8E_{c2}^2\kappa_2^2}{E_{j2}^2}\right)\sigma_x^{(2)}\sigma_x^{(1)}\right)n_p^2...\right).
\end{aligned}
$$

So the Hamiltonian of the full system can be written as:

$$
\begin{aligned}
H &= \frac{D_{33}}{2}p_C^2 + \frac{\phi_C^2}{2L} + \frac{D_{44}}{2}p_1^2 + \frac{1}{\Delta x}\sum_{i=1}^{\infty}\left(\frac{(p_{(i+1)}^p)^2}{2C_T} + \frac{(\phi_{i+1}^p - \phi_i^p)^2}{2L_T}\right) \\
&+ D_{34}p_C p_1 + E_{j1}\sigma_z^{(1)} + E_{j2}\sigma_z^{(2)} \\
&+ 4\left(\frac{E_{c1}^2\kappa_1^2}{E_{j1}}\sigma_z^{(1)} + \frac{E_{c2}^2\kappa_2^2}{E_{j2}}\sigma_z^{(2)}\right)(p_C + p_1)^2 \\
&+ T\left(\frac{4E_{C1}\kappa_1}{E_{j1}}\sigma_z^{(1)}\sigma_x^{(2)} + \frac{4E_{C2}\kappa_2}{E_{j2}}\sigma_z^{(2)}\sigma_x^{(1)}\right)(p_C + p_1) \\
&+ T\sigma_x^{(1)}\sigma_x^{(2)} \\
&+ O\left(T\frac{E_{Ci}^2\kappa_i^2 n_p^2}{E_{ji}^2}\right).
\end{aligned}
$$

Now lets compare the coefficients of $(p_C + p_1)$

$$
\frac{T\frac{4E_{C1}\kappa_1}{E_{j1}}}{4\frac{E_{c1}^2\kappa_1^2}{E_{j1}}} = \frac{T}{E_{c1}k_1} = \frac{D_{12}D_{11}e^2}{E_{c1}D_{13}} = \frac{2C_{m2}}{C_{qb2}} \ll 1.
$$

So the interaction term is small. We also notice that the harder we drive the oscillator (keeping $\kappa_i p_C \sim$ constant) the effect of the interaction term grow smaller. This can be understood since the physical coupling between the component is similar and making the oscillator dominant we repress the qubit - qubit interaction.

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