Country Portfolio Dynamics

Devereux and Sutherland

Presented by Judit Temesvary

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Outline of Presentation

1. Introduction

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2. General overview of methodology

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   1. Solution method for the steady state portfolio

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- Useful in studying the response of portfolio allocations to business cycles

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  - Optimal portfolio allocation is indeterminate in non-stochastic steady state
- Solution to first problem: use second-order approximations
- Solution to second problem: portfolio equilibrium endogenously determined so as to satisfy second order approximations of FOCs

General Overview of Methodology

- Portfolio holdings

$$\alpha(Z_t) \simeq \alpha(\bar{Z}) + \frac{\partial \alpha}{\partial Z}\left(Z_t - \bar{Z}\right)$$
$$\simeq \quad \bar{\alpha} \quad + \quad \underbrace{\hat{\alpha}_t}_{=\gamma' \hat{Z}_t}$$

- Samuelson (1970): to solve for portfolio holdings up to order *N* approximate the portfolio problem up to order *N*+2

- Expected excess returns

$$E_t\left[r_{xt+1}\right] = E_t\left[r_{1t+1} - r_{2t+1}\right] \simeq \underbrace{\bar{r}_x^e}_{=0} + \underbrace{\hat{r}_x^{e(1)}}_{=0} + \underbrace{\hat{r}_x^{e(2)}}_{\text{constant}} + \underbrace{\hat{r}_{xt}^{e(3)}}_{=\delta' \hat{Z}_t}$$

- Once portfolio problem is solved the first-order behaviour of all other variables can be solved in the usual way - including the first-order behaviour of realised asset prices and returns

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6. Solve (5) for eqm. portfolio

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6. Solve (5) for vector of coefficients for time variation in eqm. portfolio

Application

- Two countries and two (internationally traded) assets
- Home country produces good Y_H with price P_H
- Foreign country produces good Y_F with price P_F^*
- Home country agent preferences:

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u(C_\tau) + v(.) \right]$$

-
- C is a composite of home and foreign goods
- $v(\cdot)$ captures parts not relevant to the portfolio problem
- P : consumer price index for home agents

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- At end of each period, agents select portfolio holding to carry into next period

Portfolio First Order Conditions

For domestic agents:

$$E_t\left[u'(C_{t+1})r_{1,t+1}\right] = E_t\left[u'(C_{t+1})r_{2,t+1}\right]$$

For foreign agents:

$$E_t\left[u'(C^*_{t+1})r_{1,t+1}\right] = E_t\left[u'(C^*_{t+1})r_{2,t+1}\right]$$

From now, let: $\alpha_t = \alpha_{1,t}$ and $\alpha_{2,t} = W_t - \alpha_t$
Ignore non-portfolio equations for home and foreign agents

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- $\bar{\alpha}$ determined endogenously - point such that second-order approximations of the FOCs are satisfied around $\bar{X}$ and $\bar{\alpha}$

Finding the Portfolio Approximation Point

- Second-order approximation of the home-country FOC:

$$E_t\left[\hat{r}_{x,t+1}+\frac{1}{2}\left(\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right)-\rho\hat{C}_{t+1}\hat{r}_{x,t+1}\right]=O\left(\epsilon^3_{APPR}\right)$$

- Second-order approximation of the foreign-country FOC:

$$E_t\left[\hat{r}_{x,t+1}+\frac{1}{2}\left(\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right)-\rho\hat{C}^*_{t+1}\hat{r}_{x,t+1}\right]=O\left(\epsilon^3_{APPR}\right)$$

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- $$\begin{array}{c} E\left[\hat{r}_x\right]=-\frac{1}{2}E\left[\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right] \\ +\rho\frac{1}{2}E_t\left[\left(\hat{C}_{t+1}-\hat{C}^*_{t+1}\right)\hat{r}_{x,t+1}\right]+O\left(\epsilon^3_{APPR}\right) \end{array} \quad (2)$$

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- Need to find $\bar{\alpha}$ such that (1) holds

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  - α enters the non-portfolio solutions only through the budget constraint
  - Only $\bar{\alpha}$ enters the first-order approximation of the budget constraints
  - The portfolio excess return $\bar{\alpha}\hat{r}_{x,t+1}$ is a zero mean iid. random variable, and $E\left(\hat{r}_x\right) = 0$.

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- $\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \xi_t + O\left(\epsilon^2_{APPR}\right)$

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- We can use these equations to express the LHS terms of (1) in terms of ξ and exogenous terms

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- From state-space solution, we can extract the two terms on the LHS of (1):
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- $$\left(\hat{C}_{t+1} - \hat{C}^*_{t+1}\right) = [D_1]\,\xi_{t+1} + [D_2]_i\,[\varepsilon_{t+1}]^i + [D_3]_k\,[z_{t+1}]^k + O\left(\epsilon^2_{APPR}\right)$$

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- where the vector of state variables:
$$z'_{t+1} = \begin{bmatrix} x_t & s_{t+1} \end{bmatrix}$$

Expressing the LHS terms

- Using the fact that

$$\xi_{t+1} = \tilde{\alpha}\hat{r}_{x,t+1}$$

we get the reduced form equations:

$$\hat{r}_{x,t+1} = \left[\tilde{R}_2\right]_i \left[\varepsilon_{t+1}\right]^i + O\left(\epsilon^2_{APPR}\right)$$
$$\left(\hat{C}_{t+1} - \hat{C}^*_{t+1}\right) = \left[\tilde{D}_2\right]_i \left[\varepsilon_{t+1}\right]^i + \left[D_3\right]_k \left[z_{t+1}\right]^k + O\left(\epsilon^2_{APPR}\right)$$

...where the matrices are:

$$[\tilde{R}_2]_i = \frac{1}{1 - [R_1]\tilde{\alpha}} [R_2]_i$$

$$[\tilde{D}_2]_i = \left(\frac{[D_1]\tilde{\alpha}}{1 - [R_1]\tilde{\alpha}} [R_2]_i + [D_2]_i \right)$$

- Now we can evaluate the LHS of (1)!!!

Solving for the portfolio approximation point

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- Solving for the corresponding equilibrium portfolio:

$$\tilde{\alpha} = \frac{[D_2]_i[R_2]_j\ [\Sigma]^{i,j}}{([R_1][D_2]_i[R_2]_j - [D_1][R_2]_i[R_2]_j)\ [\Sigma]^{i,j}}$$

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- We need a third-order approximation of portfolio problem

Third-order Approximation

- Combining third-order approximations of home and foreign protfolio choice FOCs yields:

$$E_t \begin{bmatrix} -\rho\left(\hat{C}_{t+1}-\hat{C}^*_{t+1}\right)\hat{r}_{x,t+1} \\ +\frac{\rho^2}{2}\left(\hat{C}^2_{t+1}-\hat{C}^{2*}_{t+1}\right)\hat{r}_{x,t+1} \\ -\frac{\rho}{2}\left(\hat{C}_{t+1}-\hat{C}^*_{t+1}\right)\left(\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right) \end{bmatrix} = O\left(\epsilon^4_{APPR}\right) \quad (3)$$

$$E_t\left[\hat{r}_{x,t+1}\right] = E_t \begin{bmatrix} -\frac{1}{2}\left(\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right) \\ -\frac{1}{6}\left(\hat{r}^3_{1,t+1}-\hat{r}^3_{2,t+1}\right) \\ +\rho\left(\hat{C}_{t+1}+\hat{C}^*_{t+1}\right)\hat{r}_{x,t+1} \\ -\frac{\rho^2}{2}\left(\hat{C}^2_{t+1}+\hat{C}^{2*}_{t+1}\right)\hat{r}_{x,t+1} \\ +\frac{\rho}{2}\left(\hat{C}_{t+1}+\hat{C}^*_{t+1}\right)\left(\hat{r}^2_{1,t+1}-\hat{r}^2_{2,t+1}\right) \end{bmatrix} + O\left(\epsilon^4_{APPR}\right) \quad (4)$$

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- These are the third-order equivalents of (1) and (2)

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- where

$$\hat{\alpha}_t = \frac{1}{\beta\bar{Y}}(\alpha_t - \bar{\alpha}) = \frac{\alpha_t}{\beta\bar{Y}} - \tilde{\alpha}$$

Solving for time variation in portfolio decisions

- Postulate that $\hat{\alpha}_t$ is a linear function of the state variables of the model:

$$\hat{\alpha}_t = \gamma' \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \gamma' z_{t+1} = [\gamma]_k [z_{t+1}]^k$$

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- Let $\xi_t = \hat{\alpha}_t \hat{r}_{x,t+1}$

Matrix representation of second-order non-portfolio part

$$A_1 \begin{bmatrix} s_{t+1} \\ E_t\left(c_{t+1}\right) \end{bmatrix} = \tilde{A}_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + \tilde{A}_3 x_t + \tilde{A}_4 \Lambda_t + B\xi_t + O\left(\epsilon^3_{APPR}\right)$$

$$x_t = N x_{t-1} + \varepsilon_t$$

$$\Lambda_t = vech\left[\begin{bmatrix} x_t \\ s_t \\ c_t \end{bmatrix} \begin{bmatrix} x_t & s_t & c_t \end{bmatrix} \right]$$

Evaluating the LHS as before...

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$$\left(\hat{C} - \hat{C}^*\right) = \left[\tilde{D}_0\right] + \left[\tilde{D}_2\right]_i [\varepsilon]^i + \left[\tilde{D}_3\right]_k [z]^k$$

- $$+ \left[\tilde{D}_4\right]_{i,j} [\varepsilon]^i [\varepsilon]^j + \left(\begin{array}{c} \left[\tilde{D}_5\right]_{k,i} \\ + \left[\tilde{D}_1\right] \left[\tilde{R}_2\right]_i [\gamma]_k \end{array} \right) [\varepsilon]^i [z]^k \quad (5)$$

$$+ \left[\tilde{D}_6\right]_{i,j} [z]^i [z]^j + O\left(\epsilon^3_{APPR}\right)$$

Evaluating the LHS as before...

- Now extract equations from state-space solution and use result that excess return on portfolio time-variation $\tilde{\xi}_t = \hat{\alpha}_t \hat{r}_{x,t+1}$:
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- $$\begin{aligned} \hat{r}_x = E[\hat{r}_x] - \left[\tilde{R}_4\right]_{i,j} [\Sigma]^{i,j} + \left[\tilde{R}_2\right]_i [\varepsilon]^i + \left[\tilde{R}_4\right]_{i,j} [\varepsilon]^i [\varepsilon]^j \\ + \left(\left[\tilde{R}_5\right]_{k,i} + \left[\tilde{R}_1\right] [R_{\tilde{2}}]_i [\gamma]_k\right) [\varepsilon]^i [z]^k + O\left(\epsilon^3_{APPR}\right) \end{aligned} \tag{6}$$

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- $$\gamma_k = -\frac{\left(\left[\tilde{R}_2\right]_i \left[\tilde{D}_5\right]_{k,j} [\Sigma]^{i,j} + \left[\tilde{D}_2\right]_i \left[\tilde{R}_5\right]_{k,j} [\Sigma]^{i,j}\right)}{\left[\tilde{D}_1\right]\left[\tilde{R}_2\right]_i \left[\tilde{R}_2\right]_j [\Sigma]^{i,j}} + O\left(\epsilon_{APPR}\right) \quad (8)$$

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- This gives us the first-order effect of state variables on portfolio time variation

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  - Solve the non-portfolio part of the model to yield a state-space solution
  - Extract the appropriate rows from this solution to form the D and R matrices
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- Next: consider simple and specific example for illustration

Example

- One-good, two-country endowment economy

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho}$$

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- $$\begin{aligned} \log Y_t &= \zeta_Y \log Y_{t-1} + \varepsilon_{Y,t} \\ \log Y_t^* &= \zeta_Y \log Y_{t-1}^* + \varepsilon_{Y^*,t} \\ \log M_t &= \zeta_M \log M_{t-1} + \varepsilon_{M,t} \\ \log M_t^* &= \zeta_M \log M_{t-1}^* + \varepsilon_{M^*,t} \end{aligned}$$

Time invariant Covariance matrix of Innovations

$$\Sigma = \begin{bmatrix} \sigma_Y^2 & 0 & 0 & 0 \\ 0 & \sigma_Y^2 & 0 & 0 \\ 0 & 0 & \sigma_M^2 & 0 \\ 0 & 0 & 0 & \sigma_M^2 \end{bmatrix}$$

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- Resource constraint:

$$C_t + C_t^* = Y_t + Y_t^*$$

Equilibrium Portfolio

- Equilibrium bond holdings:

$$\tilde{\alpha}_B = -\tilde{\alpha}_{B^*} = -\frac{\sigma_Y^2}{2(\sigma_M^2 + \sigma_Y^2)(1 - \beta\zeta_Y)}$$

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- Now solve for time variation around eqm. bond holdings

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- $$\hat{\alpha}_t = \gamma' \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \gamma' z_{t+1} = [\gamma]_k [z_{t+1}]^k$$

Forming the appropriate matrices

$$\tilde{R}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\tilde{D}_1 = [2(1-\beta)]$$

$$\tilde{R}_5 = \mathbf{0}$$

$$\tilde{D}_5 = \begin{bmatrix} \Delta_2 & -\Delta_1 & -\Delta_1 & \Delta_1 \\ \Delta_1 & -\Delta_2 & -\Delta_1 & \Delta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta_3 & \Delta_3 + 2(1-\beta) & 0 & -2(1-\beta) \end{bmatrix}$$

The matrix elements

$$\Delta_1 = -\frac{1-\beta}{1-\beta\zeta_Y}\tilde{\alpha}_B\left\{(1-\zeta_Y)\rho\left[1-\zeta_Y(1-\beta)\beta\right]+\zeta_Y(1-\beta)\right\}$$

$$\Delta_2 = -\frac{1-\beta}{1-\beta\zeta_Y}\frac{\beta(1-\zeta_Y)^2\zeta_Y(1-\beta\rho)}{(1-\beta\zeta_Y)\left(1-\beta\zeta_Y^2\right)}+\Delta_1$$

$$\Delta_3 = -\frac{1-\beta}{1-\beta\zeta_Y}\frac{1-\beta\left[1-(1-\zeta_Y)\beta\rho\right]}{\beta}$$

Solutions for time variations in bond holdings

- $$\hat{\alpha}_{B,t} = \gamma_1 Y_t + \gamma_2 Y_t^* + \gamma_3 M_t + \gamma_4 M_t^* + \gamma_5 \hat{W}_t$$
  $$\hat{\alpha}_{B^*,t} = -\gamma_1 Y_t - \gamma_2 Y_t^* - \gamma_3 M_t - \gamma_4 M_t^* + (1 - \gamma_5)\, \hat{W}_t$$

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- where the vector of coefficients:

Solutions for time variations in bond holdings

- $$\hat{\alpha}_{B,t} = \gamma_1 Y_t + \gamma_2 Y_t^* + \gamma_3 M_t + \gamma_4 M_t^* + \gamma_5 \hat{W}_t$$
  $$\hat{\alpha}_{B^*,t} = -\gamma_1 Y_t - \gamma_2 Y_t^* - \gamma_3 M_t - \gamma_4 M_t^* + (1-\gamma_5)\,\hat{W}_t$$
- where the vector of coefficients:
- $$\gamma_1 = \gamma_2 = \tfrac{1}{2}\left(1 - \frac{(1-\zeta_Y)\left[1-\rho+(1-\beta)\,\beta\rho\zeta_Y^2\right]}{1-\beta\zeta_Y}\right)\tilde{\alpha}_B$$
  $$\zeta_3 = \gamma_4 = 0$$
  $$\gamma_5 = \tfrac{1}{2}$$

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