Country Portfolio Dynamics
Devereux and Sutherland

Presented by Judit Temesvary

April 2 2008
Outline of Presentation

1 Introduction
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2. General overview of methodology
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   1. Solution method for the steady state portfolio
3. Application
   1. Steady state portfolio
   2. Time variation in equilibrium portfolio
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Useful in studying the response of portfolio allocations to business cycles
Paper presents approximation method for computing equilibrium financial portfolios in stochastic open economy macro models. Time variation in these portfolios around equilibrium provides analytical solutions for optimal gross portfolio positions in any types of assets. Contribution: provides a model to analyze incomplete markets with multiple assets. Complication: impossible to use first order approximation methods in incomplete market models for two reasons: optimal portfolio allocation is indeterminate in first order approximation; optimal portfolio allocation is indeterminate in non-stochastic steady state. Solution to first problem: use second-order approximations; solution to second problem: portfolio equilibrium endogenously determined so as to satisfy second order approximations of FOCs.
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General Overview of Methodology

- Portfolio holdings

\[ \alpha(Z_t) = \alpha(\bar{Z}) + \frac{\partial \alpha}{\partial Z} (Z_t - \bar{Z}) \]

\[ = \bar{\alpha} + \frac{\hat{\alpha}_t}{\gamma \hat{Z}_t} \]

- Samuelson (1970): to solve for portfolio holdings up to order \( N \) approximate the portfolio problem up to order \( N+2 \)

- Expected excess returns

\[ E_t[r_{xt+1}] = E_t[r_{1t+1} - r_{2t+1}] = \bar{r}^e_x + \hat{r}^{e(1)}_x + \hat{r}^{e(2)}_x + \hat{r}^{e(3)}_x = 0 \]

\[ = 0 \]

\[ = 0 \text{ constant} \]

\[ = \delta \hat{Z}_t \]

- Once portfolio problem is solved the first-order behaviour of all other variables can be solved in the usual way - including the first-order behaviour of realised asset prices and returns
1. Separate variables into portfolio and non-portfolio variables.
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2. Take first order optimality conditions with respect to portfolio choice
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3. Take a second order Taylor series approximation of the FOCs around non-stochastic SS - only depends on first-order non-portfolio variables
Solution Method for Equilibrium Portfolio

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6. Solve (5) for eqm. portfolio
Solution Method for Time Variation in Equilibrium Portfolio

1. Builds on previous slide: do everything above, plus:

   - Take a third order Taylor series approximation of the FOCs around non-stochastic SS - depends on first and second order non-portfolio variables
   - Postulate that time variation in eqm. portfolio is a linear function of state variables - look for vector of coefficients on state variables
   - Write second-order approximation of non-portfolio variables as function of endogeneous portfolio eqm.
   - Use (4) to write (2) as function of time variation in eqm. portfolio and exogenous innovations only
   - Solve (5) for vector of coefficients for time variation in eqm. portfolio
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Two countries and two (internationally traded) assets
Home country produces good $Y_H$ with price $P_H$
Foreign country produces good $Y_F$ with price $P_F^*$
Home country agent preferences:

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} [u(C_{\tau}) + \nu(.)]$$

C is a composite of home and foreign goods
\(\nu (\cdot)\) captures parts not relevant to the portfolio problem
\(P\) : consumer price index for home agents
Two assets and vector of two returns (from t-1 to t):
Two assets and vector of two returns (from $t-1$ to $t$):

$$r'_t = \begin{bmatrix} r_{1,t} & r_{2,t} \end{bmatrix}$$
Two assets and vector of two returns (from t-1 to t):

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Asset payoffs and prices measured in terms of C
Two assets and vector of two returns (from t-1 to t):

$$r_t' = \begin{bmatrix} r_{1,t} & r_{2,t} \end{bmatrix}$$

Asset payoffs and prices measured in terms of C

Budget constraint for home agents:
Two assets and vector of two returns (from t-1 to t):

\[ r'_t = \begin{bmatrix} r_{1,t} & r_{2,t} \end{bmatrix} \]

Asset payoffs and prices measured in terms of C

Budget constraint for home agents:

\[ W_t = \alpha_{1,t-1}r_{1,t} + \alpha_{2,t-1}r_{2,t} + Y_t - C_t \]
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Then we have:
 Assets

- Two assets and vector of two returns (from t-1 to t):
  \[ r_t' = \begin{bmatrix} r_{1,t} & r_{2,t} \end{bmatrix} \]

- Asset payoffs and prices measured in terms of C

- Budget constraint for home agents:
  \[ W_t = \alpha_{1,t-1} r_{1,t} + \alpha_{2,t-1} r_{2,t} + Y_t - C_t \]

- Then we have:
  \[ \alpha_{1,t-1} + \alpha_{2,t-1} = W_{t-1} \]
Rewrite budget constraint in terms of excess returns on assets for home agents:
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• For foreign agents:
Rewrite budget constraint in terms of excess returns on assets for home agents:

\[ W_t = \alpha_{1,t-1} r_{x,t} + r_{2,t} W_{t-1} + Y_t - C_t \]

For foreign agents:

\[ W^*_t = \alpha^*_{1,t-1} r_{x,t} + r_{2,t} W^*_{t-1} + Y^*_t - C^*_t \]
• Rewrite budget constraint in terms of excess returns on assets for home agents:

\[
W_t = \alpha_{1,t-1} r_{x,t} + r_{2,t} W_{t-1} + Y_t - C_t
\]

• For foreign agents:

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W^*_{t} = \alpha^*_{1,t-1} r_{x,t} + r_{2,t} W^*_{t-1} + Y^*_t - C^*_t
\]

• Excess returns:
• Rewrite budget constraint in terms of excess returns on assets for home agents:

\[ W_t = \alpha_{1,t-1} r_{x,t} + r_{2,t} W_{t-1} + Y_t - C_t \]

• For foreign agents:

\[ W^*_t = \alpha^*_{1,t-1} r_{x,t} + r^*_{2,t} W^*_{t-1} + Y^*_t - C^*_t \]

• Excess returns:

\[ r_{x,t} = r_{1,t} - r_{2,t} \]
Rewrite budget constraint in terms of excess returns on assets for home agents:

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Excess returns:

\[ r_{x,t} = r_{1,t} - r_{2,t} \]

At end of each period, agents select portfolio holding to carry into next period
Portfolio First Order Conditions

For domestic agents:

\[ E_t [u'(C_{t+1})r_{1,t+1}] = E_t [u'(C_{t+1})r_{2,t+1}] \]

For foreign agents:

\[ E_t [u'(C^*_{t+1})r_{1,t+1}] = E_t [u'(C^*_{t+1})r_{2,t+1}] \]

From now, let: \( \alpha_t = \alpha_{1,t} \) and \( \alpha_{2,t} = W_t - \alpha_t \)

Ignore non-portfolio equations for home and foreign agents
Approximate around $\bar{X}$ (non-portfolio variables $\bar{C}, \bar{r}_x, \bar{Y}, \bar{W}$) and $\bar{\alpha}$ (portfolio holdings)
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$\bar{X}$ determined by symmetric non-stochastic steady state
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$$\bar{W} = 0, \bar{Y} = \bar{C}, \quad \bar{r}_x = 0$$

$$\bar{r}_1 = \bar{r}_2 = 1/\beta$$

$$\bar{\alpha}_2 = -\bar{\alpha}_1 = \bar{\alpha}$$
Approximate around \( \bar{X} \) (non-portfolio variables \( \bar{C}, \bar{r}_x, \bar{Y}, \bar{W} \)) and \( \bar{\alpha} \) (portfolio holdings)

\( \bar{X} \) determined by symmetric non-stochastic steady state

\[
\begin{align*}
\bar{W} &= 0, \quad \bar{Y} = \bar{C} \\
\bar{r}_x &= 0 \\
\bar{r}_1 &= \bar{r}_2 = 1/\beta \\
\bar{\alpha}_2 &= -\bar{\alpha}_1 = \bar{\alpha}
\end{align*}
\]

\( \bar{\alpha} \) determined endogenously - point such that second-order approximations of the FOCs are satisfied around \( \bar{X} \) and \( \bar{\alpha} \)
Finding the Portfolio Approximation Point

- Second-order approximation of the home-country FOC:
  \[
  E_t \left[ \hat{r}_{x,t+1} + \frac{1}{2} (\hat{r}_{1,t+1} - \hat{r}_{2,t+1}) - \rho \hat{C}_{t+1} \hat{r}_{x,t+1} \right] = O(\epsilon^3_{APPR})
  \]

- Second-order approximation of the foreign-country FOC:
  \[
  E_t \left[ \hat{r}_{x,t+1} + \frac{1}{2} (\hat{r}_{1,t+1} - \hat{r}_{2,t+1}) - \rho \hat{C}^*_t \hat{r}_{x,t+1} \right] = O(\epsilon^3_{APPR})
  \]
Combining the above, we get:

Equation for equilibrium portfolio holdings:

\[ E^t \hat{C}^t + 1 + \hat{C}^{t+1} + \hat{r}_x^t = 0 + O_{\epsilon^{3 \text{APPR}}} \]

Exuilibrium expected excess returns:

\[ E[\hat{r}_x^t] = \frac{1}{2} E[\hat{r}_x^{t+1}] + \rho \frac{1}{2} E^t \hat{C}^t + 1 + \hat{C}^{t+1} + \hat{r}_x^t + O_{\epsilon^{3 \text{APPR}}} \]

Need to find \( \bar{\alpha} \) such that (1) holds.
Equilibrium conditions

- Combining the above, we get:
- Equation for equilibrium portfolio holdings:
Equilibrium conditions

- Combining the above, we get:
- Equation for equilibrium portfolio holdings:
  \[ E_t \left[ (\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right] = 0 + O(\epsilon_{APPR}^3) \]
Equilibrium conditions

- Combining the above, we get:
- Equation for equilibrium portfolio holdings:
  \[ E_t \left[ (\hat{C}_{t+1} - \hat{C}^*_t) \hat{r}_{x,t+1} \right] = 0 + O \left( \epsilon^3_{APPR} \right) \quad (1) \]
- Equilibrium expected excess returns:
Equilibrium conditions

- Combining the above, we get:
- Equation for equilibrium portfolio holdings:
  \[ E_t \left[ (\hat{C}_{t+1} - \hat{C}^*_{t+1}) \hat{r}_{x,t+1} \right] = 0 + O(\epsilon^3_{APPR}) \] (1)
- Exuilibrium expected excess returns:
  \[ E[\hat{r}_x] = -\frac{1}{2} E \left[ \hat{r}^2_{1,t+1} - \hat{r}^2_{2,t+1} \right] + \rho \frac{1}{2} E_t \left[ (\hat{C}_{t+1} - \hat{C}^*_{t+1}) \hat{r}_{x,t+1} \right] + O(\epsilon^3_{APPR}) \] (2)
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Devereux and Sutherland (2006) show that three results hold:
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- $\alpha$ enters the non-portfolio solutions only through the budget constraint

The portfolio excess return $\hat{r}_{x,t+1}$ is a zero mean iid. random variable, and $E(\hat{r}_{x,t}) = 0$. 
Devereux and Sutherland (2006) show that three results hold:

- $\alpha$ enters the non-portfolio solutions only through the budget constraint
- Only $\tilde{\alpha}$ enters the first-order approximation of the budget constraints
Devereux and Sutherland (2006) show that three results hold:

- \( \alpha \) enters the non-portfolio solutions only through the budget constraint.
- Only \( \bar{\alpha} \) enters the first-order approximation of the budget constraints.
- The portfolio excess return \( \bar{\alpha} \hat{r}_{X,t+1} \) is a zero mean iid. random variable, and \( E(\hat{r}_X) = 0 \).
Let

$$\tilde{\alpha} \equiv \tilde{\alpha}/(\beta \bar{Y})$$
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Let \( \zeta_t = \tilde{\alpha} \tilde{r}_{x,t+1} \), a zero mean iid. random variable
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\[ \tilde{\alpha} = \frac{\tilde{\alpha}}{(\beta \bar{Y})} \]

Let \( \zeta_t = \tilde{\alpha} \hat{r}_{x,t+1} \), a zero mean iid. random variable

Using this in the home budget constraint approximation:
Let

\[ \tilde{\alpha} \equiv \tilde{\alpha}/(\beta \bar{Y}) \]

Let \( \tilde{\zeta}_t = \tilde{\alpha} \tilde{r}_{x,t+1} \), a zero mean iid. random variable.

Using this in the home budget constraint approximation:

\[ \hat{W}_t = \frac{1}{\beta} \hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \tilde{\zeta}_t + O(\epsilon^2_{APPR}) \]
Non-portfolio part of model

- Summarize non-portfolio side of model as:

$$\text{st}_{t+1} E t [c_{t+1}] = \text{st}_{c_t} + A_{2} x_{t} + B_{3} \xi_{t} + \epsilon_{2}$$

$$\text{app}_{x_{t}} = N x_{t,1} + \epsilon_{t}$$

- Vector of predetermined variables
- Vector of jump variables
- Vector of exogenous forcing processes
- Vector of iid. shocks
Summarize non-portfolio side of model as:

\[
A_1 \begin{bmatrix} s_{t+1} \\ E_t[c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B \xi_t + O(\epsilon_{APPR}^2)
\]

\[
x_t = N x_{t-1} + \epsilon_t
\]
Summarize non-portfolio side of model as:

\[ A_1 \begin{bmatrix} s_{t+1} \\ E_t[c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B^\xi_t + O \left( \epsilon^2_{APPR} \right) \]

\[ x_t = Nx_{t-1} + \epsilon_t \]

s: vector of predetermined variables
Summarize non-portfolio side of model as:

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A_1 \begin{bmatrix} s_{t+1} \\ E_t[c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B \xi_t + O(\epsilon_{APP}^2)
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- **s**: vector of predetermined variables
- **c**: vector of jump variables
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\[ x_t = Nx_{t-1} + \epsilon_t \]

- s: vector of predetermined variables
- c: vector of jump variables
- x: vector of exogenous forcing processes
Non-portfolio part of model

- Summarize non-portfolio side of model as:

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A_1 \begin{bmatrix} s_{t+1} \\
E_t [c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\
c_t \end{bmatrix} + A_3 x_t + B_\zeta t + O \left( \epsilon^2_{APPR} \right)
\]

\[
x_t = Nx_{t-1} + \epsilon_t
\]

- s: vector of predetermined variables
- c: vector of jump variables
- x: vector of exogenous forcing processes
- \(\epsilon\): vector of iid. shocks
The state-space solution then becomes:

\[ s_{t+1} = F_1 x_t + F_2 s_t + F_3 \xi_t + O \epsilon_{APPR} \]

\[ c_t = P_1 x_t + P_2 s_t + P_3 \xi_t + O \epsilon_{APPR} \]

We can use these equations to express the LHS terms of (1) in terms of \( \xi \) and exogenous terms.
The state-space solution then becomes:

\[ s_{t+1} = F_1 x_t + F_2 s_t + F_3 \xi_t + O(\epsilon_{APPR}^2) \]

\[ c_t = P_1 x_t + P_2 s_t + P_3 \xi_t + O(\epsilon_{APPR}^2) \]
The state-space solution then becomes:

\[ s_{t+1} = F_1 x_t + F_2 s_t + F_3 \xi_t + O \left( \epsilon^2_{APPR} \right) \]
\[ c_t = P_1 x_t + P_2 s_t + P_3 \xi_t + O \left( \epsilon^2_{APPR} \right) \]

We can use these equations to express the LHS terms of (1) in terms of \( \xi \) and exogenous terms.
Expressing the LHS terms

- From state-space solution, we can extract the two terms on the LHS of (1):
Expressing the LHS terms

- From state-space solution, we can extract the two terms on the LHS of (1):
  
  \[ \hat{r}_{x,t+1} = [R_1] \tilde{\xi}_{t+1} + [R_2]i \varepsilon_{t+1}i + O(\varepsilon^2_{APPR}) \]
Expressing the LHS terms

- From state-space solution, we can extract the two terms on the LHS of (1):
  \[ \hat{r}_{x,t+1} = [R_1] \xi_{t+1} + [R_2]_i [\varepsilon_{t+1}]^i + O(\varepsilon_{APPR}^2) \]
  \[ (\hat{C}_{t+1} - \hat{C}_{t+1}^*) = [D_1] \xi_{t+1} + [D_2]_i [\varepsilon_{t+1}]^i + [D_3]_k [z_{t+1}]^k + O(\varepsilon_{APPR}^2) \]
Expressing the LHS terms

- From state-space solution, we can extract the two terms on the LHS of (1):
  \[ \hat{r}_{x,t+1} = [R_1] \xi_{t+1} + [R_2]_i \epsilon_{t+1}^i + O(\epsilon^2_{APPR}) \]
  
  \[ (\hat{C}_{t+1} - \hat{C}_{t+1}^*) = [D_1] \xi_{t+1} + [D_2]_i \epsilon_{t+1}^i + [D_3]_k [z_{t+1}]^k + O(\epsilon^2_{APPR}) \]

- where the vector of state variables:
  \[ z'_{t+1} = \begin{bmatrix} x_t & s_{t+1} \end{bmatrix} \]
Using the fact that

\[ \xi_{t+1} = \tilde{\alpha} \hat{r}_{x,t+1} \]

we get the reduced form equations:

\[ \hat{r}_{x,t+1} = [\tilde{R}_2]_i [\epsilon_{t+1}]_i' + O \left( \epsilon_{APPR}^2 \right) \]

\[ (\hat{C}_{t+1} - \hat{C}^*_t) = [\tilde{D}_2]_i [\epsilon_{t+1}]_i' + [D_3]_k [z_{t+1}]_k^{\prime} + O \left( \epsilon_{APPR}^2 \right) \]
...where the matrices are:

\[
\begin{align*}
[\tilde{R}_2]_i &= \frac{1}{1 - [R_1] \tilde{\alpha}} [R_2]_i \\
[\tilde{D}_2]_i &= \left( \frac{[D_1] \tilde{\alpha}}{1 - [R_1] \tilde{\alpha}} [R_2]_i + [D_2]_i \right)
\end{align*}
\]

Now we can evaluate the LHS of (1)!!!
Combining the above terms, we can rewrite and simplify (1) as:

\[
[D_2]_i [R_2]_j [\Sigma]^{i,j} = 0
\]
Combining the above terms, we can rewrite and simplify (1) as:

$$[\tilde{D}_2]_i [\tilde{R}_2]_j [\Sigma]^{i,j} = 0$$

Solving for the corresponding equilibrium portfolio:

$$\tilde{\alpha} = \frac{[D_2]_i [R_2]_j [\Sigma]^{i,j}}{([R_1][D_2]_i [R_2]_j - [D_1][R_2]_i [R_2]_j) [\Sigma]^{i,j}}$$
The above solution is non time-varying: now want to analyze portfolio behavior around equilibrium.
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State variables change over time: portfolio choice problem is different in every period.
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$\alpha_t$ generally varies around $\bar{\alpha}$.
The above solution is non time-varying: now want to analyze portfolio behavior around equilibrium. State variables change over time: portfolio choice problem is different in every period. $\alpha_t$ generally varies around $\bar{\alpha}$. We want to know how risk characteristics are affected by evolution of state variables.
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We want to know how risk characteristics are affected by evolution of state variables.

Must know first-order effect of state variables on second moments of portfolio choice.
The above solution is non time-varying: now want to analyze portfolio behavior around equilibrium.

State variables change over time: portfolio choice problem is different in every period.

$\alpha_t$ generally varies around $\bar{\alpha}$

We want to know how risk characteristics are affected by evolution of state variables.

Must know first-order effect of state variables on second moments of portfolio choice.

We need a third-order approximation of portfolio problem.
Combining third-order approximations of home and foreign portfolio choice FOCs yields:

\[
E_t \begin{bmatrix}
-\rho \left( \hat{C}_{t+1} - \hat{C}_{t+1}^* \right) \hat{r}_{x,t+1} \\
+ \frac{\rho^2}{2} \left( \hat{C}_{t+1}^2 - \hat{C}_{t+1}^{2*} \right) \hat{r}_{x,t+1} \\
- \frac{\rho}{2} \left( \hat{C}_{t+1} - \hat{C}_{t+1}^* \right) \left( \hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2 \right)
\end{bmatrix} = O \left( \epsilon^4_{APPR} \right) \quad (3)
\]

\[
E_t \left[ \hat{r}_{x,t+1} \right] = E_t \begin{bmatrix}
-\frac{1}{2} \left( \hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2 \right) \\
- \frac{1}{6} \left( \hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3 \right) \\
+ \rho \left( \hat{C}_{t+1} + \hat{C}_{t+1}^* \right) \hat{r}_{x,t+1} \\
- \frac{\rho^2}{2} \left( \hat{C}_{t+1} + \hat{C}_{t+1}^* \right) \hat{r}_{x,t+1} \\
+ \frac{\rho}{2} \left( \hat{C}_{t+1} + \hat{C}_{t+1}^* \right) \left( \hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2 \right)
\end{bmatrix} + O \left( \epsilon^4_{APPR} \right) \quad (4)
\]
Combining third-order approximations of home and foreign portfolio choice FOCs yields:

\[
E_t \begin{bmatrix}
-\rho (\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \\
+\frac{\rho^2}{2} (\hat{C}_{t+1}^2 - \hat{C}_{t+1}^{2*}) \hat{r}_{x,t+1} \\
-\frac{\rho}{2} (\hat{C}_{t+1} - \hat{C}_{t+1}^*) (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2)
\end{bmatrix} = O(\epsilon_{APPR}^4) \quad (3)
\]

\[
E_t [\hat{r}_{x,t+1}] = E_t \begin{bmatrix}
-\frac{1}{2} (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \\
-\frac{1}{6} (\hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3) \\
+\rho (\hat{C}_{t+1} + \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \\
-\frac{\rho^2}{2} (\hat{C}_{t+1}^2 + \hat{C}_{t+1}^{2*}) \hat{r}_{x,t+1} \\
+\frac{\rho}{2} (\hat{C}_{t+1} + \hat{C}_{t+1}^*) (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2)
\end{bmatrix} + O(\epsilon_{APPR}^4) \quad (4)
\]

These are the third-order equivalents of (1) and (2)
GOAL: find the time variation in portfolio decisions $\hat{\alpha}_t$ such that (3) holds

\[
\hat{W}_t^{t+1} = 1 + \beta \hat{W}_t^{t+1} + \hat{Y}_t^{t+1} + \hat{C}_t^{t+1} + \tilde{\alpha} \hat{r}_x^{t+1} + \frac{1}{2} \hat{Y}_2^{t+1} + \frac{1}{2} \hat{C}_2^{t+1} + \tilde{\alpha} \hat{r}_2^{t+1} + \hat{W}_t^{t+1} \hat{r}_2^{t+1} + O(\epsilon^3)_{\text{APPR}}
\]
The Portfolio Solution

- GOAL: find the time variation in portfolio decisions $\hat{\alpha}_t$ such that (3) holds
- Apply previous procedure at a higher order
The Portfolio Solution

- GOAL: find the time variation in portfolio decisions \( \hat{\alpha}_t \) such that (3) holds
- Apply previous procedure at a higher order
- Find second-order approximation of budget constraint
The Portfolio Solution

- **GOAL**: find the time variation in portfolio decisions \( \hat{\alpha}_t \) such that (3) holds
- Apply previous procedure at a higher order
- Find second-order approximation of budget constraint

\[
\hat{W}_{t+1} = \frac{1}{\beta} \hat{W}_t + \hat{\gamma}_{t+1} - \hat{C}_{t+1} + \hat{\alpha}_t \hat{r}_{x,t+1} + \frac{1}{2} \hat{\gamma}_t^2 \\
- \frac{1}{2} \hat{C}_{t+1}^2 + \frac{1}{2} \hat{\alpha} \left( \hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2 \right) + \hat{\alpha}_t \hat{r}_{x,t+1} + \frac{1}{\beta} \hat{W}_t \hat{r}_{2,t} + O(\epsilon^3_{APPR})
\]
GOAL: find the time variation in portfolio decisions $\hat{\alpha}_t$ such that (3) holds

Apply previous procedure at a higher order

Find second-order approximation of budget constraint

$$\hat{W}_{t+1} = \frac{1}{\beta} \hat{W}_t + \hat{Y}_{t+1} - \hat{C}_{t+1} + \tilde{\alpha}_t \hat{r}_{x,t+1} + \frac{1}{2} \hat{Y}_{t+1}^2$$

$$- \frac{1}{2} \hat{C}_{t+1}^2 + \frac{1}{2} \tilde{\alpha}_t \left( \hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2 \right) + \hat{\alpha}_t \hat{r}_{x,t+1} + \frac{1}{\beta} \hat{W}_t \hat{r}_{2,t} + O (\epsilon_{APPR}^3)$$

where

$$\hat{\alpha}_t = \frac{1}{\beta Y} (\alpha_t - \bar{\alpha}) = \frac{\alpha_t}{\beta Y} - \tilde{\alpha}_t$$
Postulate that $\hat{\alpha}_t$ is a linear function of the state variables of the model:

$$\hat{\alpha}_t = \gamma' \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \gamma' z_{t+1} = [\gamma] k [z_{t+1}]^k$$
Postulate that $\hat{\alpha}_t$ is a linear function of the state variables of the model:

$$\hat{\alpha}_t = \gamma' \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \gamma' z_{t+1} = [\gamma]_k [z_{t+1}]^k$$

Now we need to find the vector of coefficients $\gamma$
Postulate that $\hat{\alpha}_t$ is a linear function of the state variables of the model:

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Now we need to find the vector of coefficients $\gamma$

Let $\zeta_t = \hat{\alpha}_t \hat{r}_{x,t+1}$
Matrix representation of second-order non-portfolio part

\[
A_1 \begin{bmatrix}
  s_{t+1} \\
  E_t(c_{t+1})
\end{bmatrix} = \tilde{A}_2 \begin{bmatrix}
  s_t \\
  c_t
\end{bmatrix} + \tilde{A}_3 x_t + \tilde{A}_4 \Lambda_t + B\xi_t + O(\epsilon^3_{APPR})
\]

\[
x_t = Nx_{t-1} + \epsilon_t
\]

\[
\Lambda_t = \text{vech} \begin{bmatrix}
  x_t \\
  s_t \\
  c_t
\end{bmatrix}
\]
Now extract equations from state-space solution and use result that excess return on portfolio time-variation $\xi_t = \hat{\alpha}_t \hat{r}_{x,t+1}$:
Evaluating the LHS as before...

- Now extract equations from state-space solution and use result that excess return on portfolio time-variation $\xi_t = \hat{\alpha}_t \hat{r}_{x,t+1}$:

  \[
  \begin{align*}
  (\hat{C} - \hat{C}^*) &= [\tilde{D}_0] + [\tilde{D}_2]_j [\varepsilon]^i + [\tilde{D}_3]_k [z]^k \\
  &+ [\tilde{D}_4]_i,j [\varepsilon]^i [\varepsilon]^j + \left( [\tilde{D}_5]_k,i [\varepsilon]^i [\varepsilon]^j \right) [\varepsilon]^i [z]^k \\
  &+ [\tilde{D}_6]_i,j [z]^i [z]^j + O(\varepsilon_{APPR}^3)
  \end{align*}
  \]
Evaluating the LHS as before...

- Now extract equations from state-space solution and use result that excess return on portfolio time-variation $\xi_t = \hat{\alpha}_t \hat{r}_{x, t+1}$:

\[
\begin{align*}
(\hat{C} - \hat{C}^*) &= [\tilde{D}_0] + [\tilde{D}_2]_i [\varepsilon]^i + [\tilde{D}_3]_k [z]^k \\
&+ [\tilde{D}_4]_{i,j} [\varepsilon]^i [\varepsilon]^j + \left( + [\tilde{D}_5]_k [\varepsilon]^i \right) [\varepsilon]^i [z]^k \\
&+ O(\varepsilon^3_{APPR})
\end{align*}
\]

\[
\hat{r}_x = E[\hat{r}_x] - [\tilde{R}_4]_{i,j} [\Sigma]^i [\varepsilon]^j + [\tilde{R}_2]_i [\varepsilon]^i + [\tilde{R}_4]_{i,j} [\varepsilon]^i [\varepsilon]^j \\
+ \left( + [\tilde{R}_5]_k [\Sigma]_k + [\tilde{R}_1] [\Sigma]_k [\gamma]_k \right) [\varepsilon]^i [z]^k + O(\varepsilon^3_{APPR})
\]
Evaluating the LHS

From state-space representation, we can also extract these equations:
Evaluating the LHS

From state-space representation, we can also extract these equations:

\[
\hat{C} = [\tilde{C}_2^H]_i [\varepsilon]^i + [\tilde{C}_3^H]_k [z]^k + O (\epsilon_{APPR}^2)
\]

\[
\hat{C}^* = [\tilde{C}_2^F]_i [\varepsilon]^i + [\tilde{C}_3^F]_k [z]^k + O (\epsilon_{APPR}^2)
\]

\[
\hat{r}_1 = [\tilde{R}_2^1]_i [\varepsilon]^i + [\tilde{R}_3^1]_k [z]^k + O (\epsilon_{APPR}^2)
\]

\[
\hat{r}_2 = [\tilde{R}_2^2]_i [\varepsilon]^i + [\tilde{R}_3^2]_k [z]^k + O (\epsilon_{APPR}^2)
\]
Combining (5), (6) and (7) and simplifying, we can rewrite and simplify (3) as:

\[ \tilde{R}_2^i \tilde{D}^k_{5, j} + \tilde{D}^1_{1, \tilde{R}^2_{j}} \left[ \gamma \right]_{k} \left[ \Sigma \right]_{i, j} + \tilde{D}^2_{i, \tilde{R}^5_{k, j}} \left[ \Sigma \right]_{i, j} = O(\epsilon_{APPR}) \]
Combining (5), (6) and (7) and simplifying, we can rewrite and simplify (3) as:

\[
\left[ \tilde{R}_2 \right]_i \left( \left[ \tilde{D}_5 \right]_{k,j} + \left[ \tilde{D}_1 \right] \left[ \tilde{R}_2 \right]_j \left[ \gamma \right]_k \right) \left[ \Sigma \right]^{i,j}
\]

\[
+ \left[ \tilde{D}_2 \right]_i \left[ \tilde{R}_5 \right]_{k,j} \left[ \Sigma \right]^{i,j} = O \left( \epsilon_{APPR}^3 \right)
\]
Combining (5), (6) and (7) and simplifying, we can rewrite and simplify (3) as:

\[
[\tilde{R}_2]_i \left( [\tilde{D}_5]_{k,j} + [\tilde{D}_1] [\tilde{R}_2]_j [\gamma]_k \right) [\Sigma]^{i,j} \\
+ [\tilde{D}_2]_i [\tilde{R}_5]_{k,j} [\Sigma]^{i,j} = O (\epsilon_{AAPPR}^3)
\]

Solving for the equilibrium coefficient vector \( \gamma \):
Combining (5), (6) and (7) and simplifying, we can rewrite and simplify (3) as:

\[
\begin{align*}
[\tilde{R}_2]_i \left( \left[ \tilde{D}_5 \right]_{k,j} + \left[ \tilde{D}_1 \right] \left[ \tilde{R}_2 \right]_j \left[ \gamma \right]_k \right) [\Sigma]^{i,j} \\
+ \left[ \tilde{D}_2 \right]_i \left[ \tilde{R}_5 \right]_{k,j} [\Sigma]^{i,j} &= O (\epsilon_A^{3}) \\
\end{align*}
\]

Solving for the equilibrium coefficient vector \( \gamma \):

\[
\gamma_k = - \left( \frac{\left[ \tilde{R}_2 \right]_i \left[ \tilde{D}_5 \right]_{k,j} [\Sigma]^{i,j} + \left[ \tilde{D}_2 \right]_i \left[ \tilde{R}_5 \right]_{k,j} [\Sigma]^{i,j}}{\left[ \tilde{D}_1 \right] \left[ \tilde{R}_2 \right]_i \left[ \tilde{R}_2 \right]_j [\Sigma]^{i,j}} \right) + O (\epsilon_A^{3}) (8)
\]
Combining (5), (6) and (7) and simplifying, we can rewrite and simplify (3) as:

\[
\hat{R}_2 \cdot \left( \hat{D}_5 \cdot k,j + \hat{D}_1 \cdot \hat{R}_2 \cdot j \cdot \gamma \cdot k \right) \cdot \Sigma \cdot i,j
\]

\[
+ \hat{D}_2 \cdot i \cdot \hat{R}_5 \cdot k,j \cdot \Sigma \cdot i,j = O(\epsilon_A^{APP})
\]

Solving for the equilibrium coefficient vector \( \gamma \):

\[
\gamma_k = - \frac{\left( \hat{R}_2 \cdot i \cdot \hat{D}_5 \cdot k,j \cdot \Sigma \cdot i,j + \hat{D}_2 \cdot i \cdot \hat{R}_5 \cdot k,j \cdot \Sigma \cdot i,j \right)}{\hat{D}_1 \cdot \hat{R}_2 \cdot i \cdot \hat{R}_2 \cdot j \cdot \Sigma \cdot i,j} + O(\epsilon_A^{APP})
\]  

(8)

This gives us the first-order effect of state variables on portfolio time variation.
To implement, all we need to do is:

1. Solve the non-portfolio part of the model to yield a state-space solution.
2. Extract the appropriate rows from this solution to form the $D$ and $R$ matrices.
3. Calculate $\gamma$ based on equation (8).
To implement, all we need to do is:

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\[ \gamma \] based on equation (8)
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To implement, all we need to do is:

- Solve the non-portfolio part of the model to yield a state-space solution
- Extract the appropriate rows from this solution to form the $D$ and $R$ matrices
- Calculate $\gamma$ based on equation (8)

Next: consider simple and specific example for illustration
Example

- One-good, two-country endowment economy

\[
U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1 - \rho}
\]
- One-good, two-country endowment economy
- Agents have CRRA preferences:

\[
U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho}
\]
Example

- One-good, two-country endowment economy
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\[ U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho} \]

- Endowments of single good and the money supply follow AR1 processes with iid. and symmetric shocks:
Example

- One-good, two-country endowment economy
- Agents have CRRA preferences:

\[
U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho}
\]

- Endowments of single good and the money supply follow AR1 processes with iid. and symmetric shocks:

\[
\begin{align*}
\log Y_t &= \zeta_Y \log Y_{t-1} + \varepsilon_{Y,t} \\
\log Y^*_t &= \zeta_Y \log Y^*_{t-1} + \varepsilon_{Y^*,t} \\
\log M_t &= \zeta_M \log M_{t-1} + \varepsilon_{M,t} \\
\log M^*_t &= \zeta_M \log M^*_{t-1} + \varepsilon_{M^*,t}
\end{align*}
\]
Time invariant Covariance matrix of Innovations

\[ \Sigma = \begin{bmatrix}
\sigma^2_Y & 0 & 0 & 0 \\
0 & \sigma^2_Y & 0 & 0 \\
0 & 0 & \sigma^2_M & 0 \\
0 & 0 & 0 & \sigma^2_M \\
\end{bmatrix} \]
Home and foreign nominal bonds are traded: $\alpha_{B,t}$ and $\alpha_{B^*,t}$
Home and foreign nominal bonds are traded: $\alpha_{B,t}$ and $\alpha_{B^*,t}$

Budget constraint:

$$W_t = \alpha_{B,t-1} r_{B,t} + \alpha_{B^*,t-1} r_{B^*,t} + Y_t - C_t$$
Asset trade

- Home and foreign nominal bonds are traded: $\alpha_{B,t}$ and $\alpha_{B^*,t}$
- Budget constraint:

$$ W_t = \alpha_{B,t-1}r_{B,t} + \alpha_{B^*,t-1}r_{B^*,t} + Y_t - C_t $$

- We also have

$$ W_t = \alpha_{B,t} + \alpha_{B^*,t} $$
Equilibrium conditions

- FOCs for consumption and bond holdings:
Equilibrium conditions

- FOCs for consumption and bond holdings:

\[
C_t^{-\rho} = \beta E_t \left[ C_t^{\rho} r_B^{*,t+1} \right]
\]

\[
C_t^{*,-\rho} = \beta E_t \left[ C_t^{*\rho} r_B^{*,t+1} \right]
\]

\[
E_t \left[ C_t^{\rho} r_B^{t+1} \right] = E_t \left[ C_t^{\rho} r_B^{*t+1} \right]
\]

\[
E_t \left[ C_t^{*\rho} r_B^{t+1} \right] = E_t \left[ C_t^{*\rho} r_B^{*t+1} \right]
\]
Equilibrium conditions

- FOCs for consumption and bond holdings:
  \[ C_t^{-\rho} = \beta E_t [C_{t+1}^{\rho} r_{B^*,t+1}] \]
  \[ C_t^{*^{-\rho}} = \beta E_t [C_{t+1}^{*\rho} r_{B^*,t+1}] \]
  \[ E_t [C_{t+1}^{-\rho} r_{B,t+1}] = E_t [C_{t+1}^{-\rho} r_{B^*,t+1}] \]
  \[ E_t [C_{t+1}^{*^{-\rho}} r_{B,t+1}] = E_t [C_{t+1}^{*^{-\rho}} r_{B^*,t+1}] \]

- Resource constraint:
  \[ C_t + C_t^* = Y_t + Y_t^* \]
Equilibrium bond holdings:

\[ \tilde{\alpha}_B = -\tilde{\alpha}_{B*} = -\frac{\sigma_Y^2}{2(\sigma_M^2 + \sigma_Y^2)(1 - \beta \zeta_Y)} \]
Equilibrium Portfolio

- Equilibrium bond holdings:

\[ \tilde{\alpha}_B = -\tilde{\alpha}_{B^*} = -\frac{\sigma_Y^2}{2(\sigma_M^2 + \sigma_Y^2)(1 - \beta \zeta_Y)} \]

- Now solve for time variation around eqm. bond holdings
Time variation in bond holdings

- Recall the general formulas from above:
Recall the general formulas from above:

\[
\gamma_k = - \frac{\left( [\tilde{R}_2]_i [\tilde{D}_5]_{k,j} [\Sigma]^{i,j} + [\tilde{D}_2]_i [\tilde{R}_5]_{k,j} [\Sigma]^{i,j} \right)}{[\tilde{D}_1] [\tilde{R}_2]_i [\tilde{R}_2]_j [\Sigma]^{i,j}} + O(\epsilon_{APPR})
\]
Recall the general formulas from above:

\[
\gamma_k = \frac{- \left( \left[ \tilde{R}_2 \right]_i \left[ \tilde{D}_5 \right]_{k,j} [\Sigma]^{i,j} + [\tilde{D}_2]_i \left[ \tilde{R}_5 \right]_{k,j} [\Sigma]^{i,j} \right)}{\left[ \tilde{D}_1 \right] \left[ \tilde{R}_2 \right]_i \left[ \tilde{R}_2 \right]_{j} [\Sigma]^{i,j}} + O(\epsilon_{APPR})
\]

\[
\hat{\alpha}_t = \gamma' \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \gamma' z_{t+1} = \left[ \gamma \right]_k z_{t+1}^k
\]
Forming the appropriate matrices

\[
\tilde{R}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}
\]

\[
\tilde{D}_1 = [2(1 - \beta)]
\]

\[
\tilde{R}_5 = 0
\]

\[
\tilde{D}_5 = \begin{bmatrix}
\Delta_2 & -\Delta_1 & -\Delta_1 & \Delta_1 \\
\Delta_1 & -\Delta_2 & -\Delta_1 & \Delta_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\Delta_3 & \Delta_3 + 2(1 - \beta) & 0 & -2(1 - \beta)
\end{bmatrix}
\]
The matrix elements

\[ \Delta_1 = -\frac{1 - \beta}{1 - \beta \zeta_Y} \tilde{\alpha}_B \{ (1 - \zeta_Y) \rho [1 - \zeta_Y (1 - \beta) \beta] + \zeta_Y (1 - \beta) \} \]

\[ \Delta_2 = -\frac{1 - \beta}{1 - \beta \zeta_Y} \frac{\beta (1 - \zeta_Y)^2 \zeta_Y (1 - \beta \rho)}{(1 - \beta \zeta_Y) (1 - \beta \zeta_Y^2)} + \Delta_1 \]

\[ \Delta_3 = -\frac{1 - \beta}{1 - \beta \zeta_Y} \frac{1 - \beta}{\beta} [1 - (1 - \zeta_Y) \beta \rho] \]
Solutions for time variations in bond holdings

\[ \hat{a}_{B,t} = \gamma_1 Y_t + \gamma_2 Y_t^* + \gamma_3 M_t + \gamma_4 M_t^* + \gamma_5 \hat{W}_t \]

\[ \hat{a}_{B^*,t} = -\gamma_1 Y_t - \gamma_2 Y_t^* - \gamma_3 M_t - \gamma_4 M_t^* + (1 - \gamma_5) \hat{W}_t \]
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where the vector of coefficients:

\[ \gamma_1 = \gamma_2 = \frac{1}{2} \left( 1 - \frac{(1 - \zeta_Y) [1 - \rho + (1 - \beta) \beta \rho \zeta_Y^2]}{1 - \beta \zeta_Y} \right) \hat{\alpha}_B \]

\[ \zeta_3 = \gamma_4 = 0 \]

\[ \gamma_5 = \frac{1}{2} \]
Paper extends Devereux and Sutherland (2006) solution method for equilibrium portfolios to higher order approximations
Conclusion

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- Finds analytical expressions for dynamic behavior of portfolios in open economy GE models
- Provides simple and clear insights into factors determining the dynamic evolution of portfolios