Solution for Problem
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Topic 6: Externalities and public goods

Problem 2

(a) There are 2 goods in the economy: a private good y and a public good G. The resource constraint is

$$y_A + y_B + \frac{G^2}{2} = 5 + 3$$

Hence, the relationship between the total availability of y (y_A+y_B) and of G is given by

$$y = 8 - \frac{G^2}{2}$$

This expression is just the PPF for the economy.

(b) In order to characterize the set of Pareto efficient allocations we need to maximize the utility of one agent, say agent A, subject to holding the utility of agent B fixed, and subject to the PPF (or resource constraint which is equal to the PPF in this case). That is, we need to solve the problem:

$$\max 2\ln(G) + y_A$$

subject to

$$2\ln(G) + y_B = \overline{U}$$

and

$$y_A + y_B = 8 - \frac{G^2}{2}$$

If we substitute for y_B from the resource constraint into the first constraint, we get

$$2\ln(G) + 8 - \frac{G^2}{2} - y_A = \overline{U}$$

Now, substituting for y_A from this constraint into the objective function, yields:

$$\max 2\ln(G)+2\ln(G)-\frac{G^2}{2}+8-\overline{U}.$$

The first order condition for this problem is:

$$\frac{2}{G}+\frac{2}{G}-G=0.$$

Solving for G yields G* = 2. Hence, in a Pareto efficient allocation, G* = 2 and $y_A + y_B = 8 - 2^2/2 = 6$. Notice that there are infinitely many Pareto efficient allocations because any G* = 2 and $y_A + y_B = 6$ is Pareto efficient.

(c) To find a Walrasian equilibrium, we need to find a vector (p_G, p_y) such that the markets for G and y clear. Let's start with the agents' problems. Solving them will give is the demands for G and y. Since the agents' problems are symmetric, we can solve the problem of agent A. The problem of agent B is analogous. Here is the problem of A:

$$\max 2\ln(g_A+g_B)+y_A$$

subject to

$$p_G g_A + y_A = 5+\frac{\pi}{2}$$

To solve the problem we can substitute for y_A from the constraint into the objective function and get:

$$\max 2\ln(g_A+g_B)-p_G g_A+5+\frac{\pi}{2}$$

The first order condition for the problem is

$$\frac{2}{g_A+g_B}-p_G=0.$$

Hence, in a Walrasian equilibrium,

$$g_A+g_B\equiv G=\frac{2}{p_G}.$$

As for the firm, it chooses G to maximize it profit, so it problem is

$$\max \pi = p_G G-\frac{G^2}{2}$$

The first order condition for this problem is

$$p_G - G = 0$$

Altogether then, in a Walrasian equilibrium,

$$G = \frac{2}{p_G} = p_G \quad \Rightarrow \quad p_G = G = \sqrt{2}$$

(d) Since Pareto efficiency requires that G = 2, we underprovision of the public good relative to the Pareto efficient allocation computed in (b). The intuition is that each agent ignores the effect of the public good on the other agent, or alternatively, each agent has an incentive to free-ride on the provision of the public good by the other agent.

(e) Suppose we now have a system of subsidies, s_A and s_B given to the purchases of the public good by the two agents. With the subsidies in place, the price that A pays is p_A - s_A and the price that B pay is p_B - s_B.

Hence, in a Walrasian equilibrium with subsidies, the demand for the public good is

$$g_A + g_B \equiv G = \frac{2}{p_G - s},$$

where s = s_A = s_B. The supply of the public good is still G = p_G. We require that s will be chosen such that when demand equals supply, G will be equal to 2. But if G = 2 then p_G = 2 (this is the firm's supply). Hence, the demand can be equal to 2 given that p_G = 2 provided that s = 1.

Problem 3

(a) Since the technology is such that one unit of y can be converted to one unit of G, the PPF is given by

$$y_A + y_B + G = 10 + 8$$

(b) In order to characterize the set of Pareto efficient allocations we need to maximize the utility of one agent, say agent A, subject to holding the utility of agent B fixed, and subject to the PPF. That is, we need to solve the problem:

$$\max 5G - \frac{G^2}{2} + y_A$$

subject to

$$5G - \frac{G^2}{2} + y_B = \overline{U}$$

and

$$y_A + y_B + G = 10 + 8$$

If we substitute for y_B from the resource constraint into the first constraint and then substitute from this constraint into the objective function we get

$$\max 5G - \frac{G^2}{2} + \left(5G - \frac{G^2}{2} - G + 18 - \overline{U}\right)$$

The first order condition for this problem is:

$$5 - G + 5 - G - 1 = 0.$$

Solving for G yields G* = 4.5. Hence, in a Pareto efficient allocation, G* = 4.5 and $y_A + y_B = 18 - 4.5 = 13.5$.

(c) In a voluntary contribution game, we need to find the Nash equilibrium (z_A, z_B). To do that, we need to characterize the best response functions of the two agents. Starting with agent A, his best response against z_B is given by the solution to the problem:

$$\max 5(z_A + z_B) - \frac{(z_A + z_B)^2}{2} + \underbrace{10 - z_A}_{y_A}$$

The first order condition for this problem is

$$5 - (z_A + z_B) - 1 = 0$$

The problem of agent B, is analogous::

$$\max 5(z_A + z_B) - \frac{(z_A + z_B)^2}{2} + \underbrace{8 - z_B}_{y_B}$$

The first order condition for this problem is also given by

$$5 - (z_A + z_B) - 1 = 0$$

Since the two first order conditions are identical, the equilibrium is symmetric and in equilibrium, $z_A + z_B = 4$.

(d) The amount of G that you computed in (c) is Pareto inefficient because we have only 4 units of G instead of 4.5. Once again, we get underprovision of the public good because the agents free-ride on one another's provision of the public good and ignore each other's benefit from the public good when they make private contributions.