Solution for Review Problems
Yossi Spiegel

Problem 6

(a) The profit functions are:

$$\pi_A = W_A^{1/2} Z_A^{1/2} - p_W W_A - p_Z Z_A$$

and

$$\pi_B = \frac{W_B^{1/2} Z_B^{1/2}}{W_A} - p_W W_B - p_Z Z_B$$

(b) To find out the Walrasian equilibrium we need to find a pair of prices, p_W and p_Z (the price of the output is 1) that clear the markets for the inputs. There is a third market here - the market for the output – but we have no information about it since we do not know the consumers' demands. By the Walras law we can therefore ignore that market and only focus on the market for the two inputs.

To characterize the demands of firm A for the two inputs we need to solve the following problem:

$$\max \pi_A = W_A^{1/2} Z_A^{1/2} - p_W W_A - p_Z Z_A$$

The first order conditions for the problem are:

$$\frac{\partial \pi_A}{\partial W_A} = \frac{Z_A^{1/2}}{2W_A^{1/2}} - p_W = 0$$

and

$$\frac{\partial \pi_A}{\partial Z_A} = \frac{W_A^{1/2}}{2Z_A^{1/2}} - p_Z = 0$$

Likewise, the demands for firm B's problem, are given by the following first order conditions:

$$\frac{\partial \pi_B}{\partial W_B} = \frac{Z_B^{1/2}}{2W_A W_B^{1/2}} - p_W = 0$$

and

$$\frac{\partial \pi_B}{\partial Z_B} = \frac{W_B^{1/2}}{2W_A Z_B^{1/2}} - p_Z = 0$$

From the first order conditions for firm A's problem it follows that

$$\frac{Z_A^{1/2}}{W_A^{1/2}} = 2p_W \quad \Rightarrow \quad \frac{Z_A}{W_A} = 4p_W^2$$

and

$$\frac{W_A^{1/2}}{Z_A^{1/2}} = 2p_Z \quad \Rightarrow \quad \frac{Z_A}{W_A} = \frac{1}{4p_Z^2}$$

Hence, in a Walrasian equilibrium, it must be that

$$4p_W^2 = \frac{1}{4p_Z^2}$$

Likewise, if we use the first order conditions of firm B we get that in a in a Walrasian equilibrium, it must be that

$$4W_A^2 p_W^2 = \frac{1}{4W_A^2 p_Z^2}$$

Clearly, the last two equations can be compatible only if $W_A = 1$; otherwise the conditions for Walrasian equilibrium would not hold.

(c) Having found that in A walrasian equilibrium, $W_A = 1$, it follows from the market clearing condition for W that $W_B = 3\text{-}1 = 2$.

We can now substitute $W_A = 1$ and $W_B = 2$ into the first order conditions of the two firms and find that the demands for Z are:

$$\frac{Z_A^{1/2}}{1} = 2p_W \quad \Rightarrow \quad Z_A = 4p_W^2$$

and

$$\frac{{Z_B}^{1/2}}{2^{1/2}} = 2p_W \quad \Rightarrow \quad Z_B = 8{p_W}^2$$

Substituting these demands into the market clearing condition for Z, yields:

$$4{p_W}^2 + 8{p_W}^2 = \frac{4}{3} \quad \Rightarrow \quad p_W = \frac{1}{3}$$

e) The solution is not Pareto efficient because we have a problem with externalities. The actual computation follow our usual steps: maximize the profit of firm A subject to holding the profit of firm B fixed at some level and subject to the resource constraint.