An example of externalities - the multiplicative case

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Consider an exchange economy with two agents, A and B, who consume two goods, x and y. This economy however, differs from the usual exchange economies we've seen so far in that here, the utility of agent B is affected not only by his own consumption of goods x and y, but also by agent A's consumption of good x. Specifically, let's assume that the preferences of agents A and B over the goods x and y are represented by the following utility functions:

$$U^A(x_A, y_A) = \left(Z x_A - \frac{x_A^2}{2}\right) + y_A, \qquad U^B(x_B, y_B, x_A) = \left((Z + \gamma x_A) x_B - \frac{x_B^2}{2}\right) + y_B, \tag{1}$$

where Z is a positive parameter, while γ is either positive or negative. If $\gamma > 0$, then agent A's consumption of good x has a positive externality on agent B in the sense that it enhances agent B's utility. On the other hand, if $\gamma < 0$ then agent A's consumption of good x has a negative externality on agent B as it lowers B's utility. In any event, the externality enters the problem in a multipplicative way so it affects both the utility and marginal utility of agent B. When $\gamma = 0$, agent B is not directly affected by A's consumption of good x so the model is exactly like the usual models of exchange economies we've seen before. It should be emphasized that even when $\gamma = 0$ agent A's consumption of x affects agent B, since the more agent A consumes, the less x is left for agent B. Yet this affect is indirect in the sense that it arises through the prices of x and y rather than by directly affecting the utility of agent B.

Note that the two utility functions are quasi-linear in good y. The reason for this assumption is that it simplifies the analysis greatly and allows us to focus on the problem at hand in as simple way as possible.

Assuming that the initial endowments of the two agents are $(\bar{x}_A, \bar{y}_A)$ and $(\bar{x}_B, \bar{y}_B)$, the respective budget constraints for the two agents are given by

$$p_x x_A + p_y y_A = p_x \bar{x}_A + p_y \bar{y}_A, \qquad p_x x_B + p_y y_B = p_x \bar{x}_B + p_y \bar{y}_B. \tag{2}$$

By Walras' law, we can normalize one of the prices to 1. If we set $p_y = 1$ and substitute for y_A and y_B from the two budget constraints into equation (1), the utility functions become:

$$U^A(x_A) = \left(Z x_A - \frac{x_A^2}{2} \right) - p_x x_A + M_A,$$

(3)

$$U^B(x_B, x_A) = \left((Z + \gamma x_A) x_B - \frac{x_B^2}{2} \right) - p_x x_B + M_B,$$

where $M_A \equiv p_x \bar{x}_A + \bar{y}_A$ and $M_B = p_x \bar{x}_B + \bar{y}_B$, are the incomes of the two agents. Hence, the assumption that the utility functions are quasi-linear in y allows us to "get rid" of y and express the utility of the two agents only in terms of their consumption of x. This will help us to focus attention on the externality issue that arises after all due to the consumption of good x by agent A.

Having written down the model we now proceed to solve for the Walrasian equilibrium and to show that its is Pareto inefficient.

Walrasian equilibrium

By Walras's law the market for y automatically clears once the market for x clears. Hence, we can always normalize the price of y to 1. Hence, a Walrasian equilibrium is a price, p_x*, such that the market for x clears. In order to find p_x let us first derive the demands for x. Maximizing $U^A(x_A)$ w.r.t. x_A and $U^B(x_B,x_A)$ w.r.t x_B, we obtain the following demand functions:

$$x_A(p_x) = Z - p_x, \qquad x_B(p_x) = \left(Z + \gamma x_A\right) - p_x. \tag{4}$$

Equation (4) indicates that the two demand functions are linear and have slopes equal to 1. However, because of the externality, the intercept of the demand function of agent B depends on x_A and is either increasing or decreasing in x_A, depending on whether γ is positive or negative.

The aggregate demand for good x is therefore given by:

$$x(p_x) \equiv x_A(p_x) + x_B(p_x) = 2Z + \gamma x_A(p_x) - 2p_x. \tag{5}$$

But, since x_A = Z - p_x, we can write the aggregate demand as follows:

$$x(p_x) = (2 + \gamma)(Z - p_x). \quad \textbf{(6)}$$

Since the aggregate supply of x is $\bar{x} \equiv \bar{x}_A + \bar{x}_B$, the equilibrium price is determined by the following market clearing condition:

$$(2 + \gamma)(Z - p_x) = \bar{x}. \quad \textbf{(7)}$$

Solving for p_x, the equilibrium price is:

$$p_x^* = Z - \frac{\bar{x}}{2 + \gamma}. \quad \textbf{(8)}$$

Note that the higher is γ, the larger is the externality and the higher is the demand of agent B for x. Not surprisingly then, this increase in demand leads to an increase in p_x* to ensure that the market clears.

Finally, substituting for p_x* in equation (4) reveals that in a Walrasian equilibrium, the allocation of x is such that:

$$x_A^* = Z - p_x^* = \frac{\bar{x}}{2 + \gamma}, \qquad x_B^* = \bar{x} - x_A^* = \frac{(1 + \gamma)\bar{x}}{2 + \gamma}. \quad \textbf{(9)}$$

Again, the larger is the externality (i.e., the larger is γ), the more x is consumed by agent B and the less x is consumed by A.

Pareto efficiency

To characterize the set of Pareto efficient allocations, recall that this set is determined by the following maximization problem:

$$\begin{aligned} &\underset{x_A, x_B}{Max} \quad U^A(x_A, y_A) \\ &s.t. \quad U^B(x_B, y_B, x_A) = \bar{U}^B \\ &\qquad\; x_A + x_B = \bar{x} \\ &\qquad\; y_A + y_B = \bar{y}. \end{aligned} \tag{10}$$

Given equation (1) and substituting for x_B and y_B from the two last constraints into the maximization problem we can write this problem as follows:

$$\begin{aligned} &\underset{x_A, y_A}{Max} \quad \left(Z x_A - \frac{x_A^2}{2} \right) + y_A, \\ &s.t. \quad \left((Z + \gamma x_A)(\bar{x} - x_A) - \frac{(\bar{x} - x_A)^2}{2} \right) + (\bar{y} - y_A) = \bar{U}^B. \end{aligned} \tag{11}$$

Now, using the constraint to solve for y_A and substituting in the objective function, the maximization problem becomes:

$$\underset{x_A}{Max} \quad \left(Z x_A - \frac{x_A^2}{2} \right) + \left((Z + \gamma x_A)(\bar{x} - x_A) - \frac{(\bar{x} - x_A)^2}{2} \right) + (\bar{y} - \bar{U}^B). \tag{12}$$

Solving this problem for x_A, reveals that at a Pareto efficient allocation, we have:

$$x_A^{**} = \frac{\bar{x}}{2}. \tag{13}$$

Using this expression, it follows that:

$$x_B^{**} = \bar{x} - x_A^{**} = \frac{\bar{x}}{2}. \qquad \textbf{(14)}$$

That is, at Pareto efficient allocations, the consumption of x should be divided between the two agents equally, irrespective of the externality. It is important to note though that this result depends heavily on the assumed functional forms and need not (and indeed does not) hold in general. Still the important result here is that since the Walrasian equilibrium allocation does depend on γ, it will not be Pareto efficient unless $\gamma = 0$. That is, in the absence of an externality, the Walrasian equilibrium is Pareto efficient. However, if we have an externality whether a positive or a negative one, we no longer have Pareto efficiency.

In particularly, when the externality is positive so that $\gamma > 0$, $x_A^* < x_A^{**}$ so that a Walrasian equilibrium results in "too little" consumption of x_A relative to the efficient level. On the other hand, if the externality is negative so that $\gamma < 0$, $x_A^* > x_A^{**}$ so that a Walrasian equilibrium results in "too much" consumption of x_A relative to the efficient level. The intuition for these results is as follows: A Walrasian mechanism is a decentralized mechanism in the sense that each agent takes into account only his own utility and decides how much to consume based only on this information. Thus, when agent A considers what to do, he does not take into account how his consumption of x_A will affect agent B. If it turns out that the externality is positive, agent A fails to take into account some of the benefits from his consumption of x because it accrues to agent B. Hence, agent A will not consume as much x_A as is efficient when both the utilities of A and B are taken into account. Similarly, if the externality is negative, agent A fails to take into account the harm of his consumption of x to agent B and he will consume more than is efficient when agent B’s utility is also accounted for.

Pigouvian taxation

One way to correct the market failure and restore Pareto efficiency is to impose a tax on the consumption of x_A when there is a negative externality so that there is "too much" x_A in equilibrium (and thereby lower the consumption of x_A), or subsidize x_A when the externality is positive and there is "too little" x_A in order to encourage A to increase his consumption of x. The taxes and subsidies needed to restore Pareto efficiency in the presence of externalities are called "Pigouvian" taxes after the British economist Arthur Pigou who originally suggest these taxes.

To compute the Pigouvian taxes in our case, suppose that there is a tax t on x_A. When $t < 0$, A gets a subsidy when he consumes x. Now, agent A’s budget constraint becomes:

$$(p_x + t)x_A + p_y y_A = p_x \bar{x}_A + p_y \bar{y}_A. \quad \textbf{(15)}$$

Therefore agent A’s demand function for x_A is now given by:

$$x_A(p_x) = Z - p_x - t. \quad \textbf{(16)}$$

Agent B’s budget constraint remains as before.

The aggregate demand for good x is therefore given by:

$$x(p_x) \equiv x_A(p_x) + x_B(p_x) = 2Z + \gamma x_A(p_x) - 2p_x - t. \quad \textbf{(17)}$$

But, since $x_A = Z - p_x - t$, we can write the aggregate demand as follows:

$$x(p_x) = (2 + \gamma)(Z - p_x) - (1 + \gamma)t. \quad \textbf{(18)}$$

Since the aggregate supply of x is $\bar{x} \equiv \bar{x}_A + \bar{x}_B$, the equilibrium price is determined by the following market clearing condition:

$$(2 + \gamma)(Z - p_x) - (1 + \gamma)t = \bar{x}. \quad \textbf{(19)}$$

Solving for p_x, the equilibrium price is:

$$p_x^* = Z - \frac{\bar{x} + (1 + \gamma)t}{2 + \gamma}. \quad \textbf{(20)}$$

Substituting this expression in equation (16), reveal that given t:

$$x_A^* = Z - p_x^* - t = \frac{\bar{x} - t}{2 + \gamma}. \qquad \textbf{(21)}$$

To restore Pareto efficiency we need to choose t such that $x_A^* = \bar{x}/2$. This can be achieved by setting:

$$t^* = -\frac{\gamma \bar{x}}{2}. \qquad \textbf{(22)}$$

Equation (22) shows that if $\gamma > 0$, i.e., there is a positive externality, $t^* < 0$ so that agent A gets a subsidy to encourage him to consume more x and thereby correct the inefficiency that results from the fact that agent A consumes "too little" x. On the other hand, if $\gamma < 0$ and agent A exerts a negative externality on agent B, then $t^* > 0$ and agent A's consumption of x is taxed to induce him to cut on his consumption of x. Moreover, the tax or subsidy are increasing when γ is larger in absolute value since then the externality is more significant.

The Coase theorem

Another way to correct the externality and restore Pareto efficiency arises when the two agents can costlessly bargain with one another over the consumption of x_A. In that case, Ronald Coase, suggested that all is needed is to set up "property rights" in some way or another and let the two agent bargain freely until they reach a Pareto efficient allocation.

To see this point, suppose that the law gives agent B the exclusive property rights over the use of x so that agent B can confiscate A's endowment of x and consume all of it by himself. Moreover, suppose that agent B can approach agent A before confiscating his endowment of x and offer agent A a take-it-or-leave offer, according to which agent B will allow agent A to consume x_A units of x if in return agent A will pay agent B the amount of T dollars. If agent A refuses, the two cannot reach any agreement (in this sense B's offer is a take-it-or-leave-it offer) and agent A consumes 0 units of x while agent B consumes $\bar{x}$ units. Since the two agents do not trade x's, they cannot trade y's as well since, if agent A wants to buy y's he has no x's to pay for it and if he sells y's he cannot get anything in return since B consumes all of x on his own.

Given this setup, agent A realizes that if reject's B's offer then he will have only $\bar{y}_A$ units of y to consume in which case, his utility will be equal to $U^A(0, \bar{y}_A) = \bar{y}_A$. But, if A accepts B's offer then his utility is:

$$U^A(x_A) = \left(Z x_A - \frac{x_A^2}{2}\right) + \bar{y}_A - T. \tag{23}$$

Comparing this expression with $U^A(0, \bar{y}_A) = \bar{y}_A$, it is clear that the most that agent A will agree to pay agent B is:

$$T^* = \left(Z x_A - \frac{x_A^2}{2}\right). \tag{24}$$

Agent B of course anticipates T* so he chooses x_A to maximize his utility subject to the resources constraint. That is, agent B solves the following problem:

$$\begin{aligned} &\underset{x_A, y_A, x_B, y_B}{Max} \left((Z + \gamma x_A) x_B - \frac{x_B^2}{2}\right) + \bar{y}_B + T^* \\ &s.t. \quad \bar{x}_A + \bar{x}_B = \bar{x} \\ &\qquad\;\; \bar{y}_A + \bar{y}_B = \bar{y}. \end{aligned} \tag{25}$$

Substituting from the constraints for x_B and y_B and substituting for T* from equation (24), B's maximization problem becomes:

$$\underset{x_A}{Max} \left(Z x_A - \frac{x_A^2}{2}\right) + \left((Z + \gamma x_A)(\bar{x} - x_A) - \frac{(\bar{x} - x_A)^2}{2}\right) + \bar{y}_B. \tag{26}$$

This problem however is essentially the same as the one in (12) (the differ only by a constant that does not affect the solution). Hence, B's choice of x_A will be Pareto efficient. In other words, B will offer A to consume half of the total endowment of x in return for a payment that will extract all of A's utility from doing so (relative to A's option to consume only his own endowment of y).

Now one can check that the problem can be also solved by assigning the property rights over the use of x to agent A and allow agent A to make agent B a take-it-or-leave offer, according to which agent A will allow agent B to consume x_A units of x if in return agent B will pay agent A T dollars. If agent B rejects the offer, his utility will be equal to $U^B(0, \bar{y}_B, x_A) = \bar{y}_B$. But, if B accepts A's offer then B's utility is:

$$U^B(x_B, \bar{y}_B, x_A) = \left((Z + \gamma\, x_A)\, x_B - \frac{x_B^2}{2} \right) + \bar{y}_B - T^*. \qquad \textbf{(27)}$$

Comparing the two levels of utility, it is clear that the most that agent B will agree to pay for the right to consume x is

$$T^* = (Z + \gamma\, x_A)\, x_B - \frac{x_B^2}{2}. \qquad \textbf{(28)}$$

Anticipating T*, agent A chooses x_A to maximize his utility subject to the resources constraint. That is, agent A solves the following problem:

$$\begin{aligned} &\underset{x_A, y_A, x_B, y_B}{Max} \left(Z\, x_A - \frac{x_B^2}{2} \right) + \bar{y}_A + T^* \\ &s.t. \quad \bar{x}_A + \bar{x}_B = \bar{x} \\ &\qquad\;\; \bar{y}_A + \bar{y}_B = \bar{y}. \end{aligned} \qquad \textbf{(29)}$$

Substituting from the constraints for x_B and y_B and substituting for T* from equation (28), A's maximization problem becomes:

$$\underset{x_A}{Max} \quad \left(Z\, x_A \; - \; \frac{x_A^2}{2} \right) \; + \; \left((Z \; + \; \gamma \; x_A)\, (\bar{x} - x_A) \; - \; \frac{(\bar{x} - x_A)^2}{2} \right) \; + \; \bar{y}_A. \qquad \textbf{(30)}$$

Since this problem is identical to the one in (26) (up to a constant that has no impact on the results), it is obvious that this problem will also lead to a Pareto efficient allocation.

The conclusion then is that property rights, i.e., which way we assign the rights to use the goods, do not matter so long as the agents can freely and costlessly bargain with one another since they will find a way to reach the Pareto efficient allocations. The reason of course is that so long as the allocation is inefficient that two agents can make themselves both better-off by agreeing to reallocate resources so not surprisingly, they should always find a way to get to Pareto efficient allocations. Of course property rights do matter in that the utilities of the two agents depend on the property rights. However, the fact that the final allocation is Pareto efficient is independent of the assignment of the property rights.