# An example of externalities - the additive case

Yossi Spiegel

Consider an exchange economy with two agents, A and B, who consume two goods, x and y. The preferences of the two agents are represented by the following quasi-linear utility functions:

$$U^A(x_A, y_A) = \left(4x_A - \frac{x_A^2}{2}\right) + y_A, \qquad U^B(x_B, y_B, x_A) = \left(4x_B - \frac{x_B^2}{2}\right) + \gamma x_A + y_B, \quad \textbf{(1)}$$

where $\gamma$ is either positive or negative. If $\gamma > 0$, then agent A's consumption of good x enhances agent B's utility, while if $\gamma < 0$ agent A's consumption of good x lowers the utility of agent B. Notice that if $\gamma = 0$ there is no externality as agent B is not directly affected by A's consumption of good x. In this case, the model becomes the usual model of a 2 x 2 exchange economy. In any event note that the externality affects agent B's utility in an additive way. In particularly, it has no effect on agents B's marginal utility from either x or y. As we shall see, this assumption will simplify matters a great deal. To complete the model assume that initially, each agent is endowed with 2 units of good x. Hence, the total endowment of x in the economy is 4. Since the utility function of each agent is quasi-linear in y, it is not important for us to know the initial endowments of good y as the solutions we'll get will be independent of the exact values of these endowments (provided that these endowments are "sufficiently large").

We now solve for the Walrasian equilibrium in the economy and demonstrate that it is Pareto inefficient.

## Walrasian equilibrium

By Walrus's law we can normalize the price of y to 1 and focus only on the market for x. Once the market for x clears, the market for y will automatically clear as well. We chose to focus on the market for x because this is the "interesting" market where the externality is present (of course, by Walrus' law we might as well focus on the "less interesting" market for y). Given our choice to focus on the market for x, a Walrasian equilibrium is a price, $p_x$*, such that the market for x clears. However, instead of actually solving for the Walrasian equilibrium, solving for the Pareto efficient allocation, and comparing the two, we shall only write the conditions that determine the two allocations and show that these conditions are not the same (which implies of course that the Walrasian equilibrium will be Pareto inefficient).

We begin with the condition that defines the Walrasian equilibrium. In equilibrium, both agents maximize their respective utility functions subject to their budget constraints. Hence, in a Walrasian equilibrium, the following first order conditions must hold:

$$4 - x_A = p_x, \tag{2}$$

and

$$4 - x_B = p_x. \tag{3}$$

The left side of equations (2) and (3) are the marginal utilities of agents A and B from x. The right side of the two equations is just the price of x. Hence the equations say that each agent will buy units of x up to the point where his/her marginal utility is equal to the price of the good. Taken together, the two equations reveal that at a Walrasian equilibrium,

$$4 - x_A = 4 - x_B. \tag{4}$$

Since the marginal utilities of both agents from consuming y are equal to 1 (we did not assume quasi-linear preferences for nothing!) the left side of equation (4) is also the marginal rate of substitution for agent A and the right side of the equation is the marginal rate of substitution for agent B. That is, equation (4) states the familiar condition that says that in a Walrasian equilibrium, the marginal rates of substitution are the same for both agents.

Since feasibility requires that $x_A+x_B = 4$, the equilibrium condition can be written as:

$$4 - x_A = 4 - (4 - x_A) = x_A, \Rightarrow x_A^* = 2. \tag{5}$$

## Pareto efficiency

To characterize the set of Pareto efficient allocations, recall that this set is determined by the following maximization problem:

$$\begin{aligned} &\underset{x_A, x_B}{Max} \quad U^A(x_A, y_A), \\ &s.t. \quad U^B(x_B, y_B, x_A) = \bar{U}^B, \\ &\qquad x_A + x_B = 4, \\ &\qquad y_A + y_B = \bar{y}, \end{aligned} \qquad \textbf{(6)}$$

where $\bar{y}$ is the initial endowment of good y. Using equation (1) and substituting for $x_B$ and $y_B$ from the last two constraints into the maximization problem, we can write the problem as follows:

$$\underset{x_A}{Max} \quad \left(4\,x_A - \frac{x_A^2}{2}\right) + \left(4(\bar{x} - x_A) - \frac{(\bar{x} - x_A)^2}{2}\right) + \gamma\, x_A + (\bar{y} - \bar{U}^B). \qquad \textbf{(7)}$$

That is, the problem of finding the Pareto efficient allocation is the same as finding the optimal Benthamite allocation (i.e., the allocation that maximizes the sum of utilities which is the criterion for social optimum advocated by Jeremy Bentham). Maximizing the problem with respect to $x_A$, reveals that at a Pareto efficient allocation,

$$4 - x_A + \gamma = x_A, \quad \Rightarrow \quad x_A^{**} = \frac{4 + \gamma}{2}. \qquad \textbf{(8)}$$

To interpret the condition that defines Pareto efficiency, note that the left side of equation (8), is the marginal social benefit from $x_A$ which includes agents A's marginal utility from $x_A$ as well as the externality that A imposes on B. The right side of the equation represents the marginal utility of $x_B$. Since $x_B$ does not create an externality, the right side of equation (8) is also the marginal social benefit of $x_B$. Hence the condition for Pareto efficiency requires that the marginal social benefits are equalized.

To compare the Walrasian equilibrium allocation with the Pareto efficient allocation, note that $x_A^{**} > x_A^*$ when $\gamma > 0$ and that $x_A^{**} < x_A^*$ when $\gamma < 0$. That is, in a Walrasian equilibrium, agent A consumes "too little" x when $\gamma > 0$ but consumes "too much" x when $\gamma < 0$. The intuition for these results is that the Walrasian mechanism is a decentralized mechanism as agents take into account only their own utility when they decide how much to consume and ignore the impact of their actions on other agents. In this example, agent A fails to take into account the impact of his consumption of $x_A$ on agent B’s welfare. On the other hand, Pareto efficiency requires that the externalities will be taken into account and hence we have a market failure.

## Pigouvian taxation

One way to correct the market failure that arises because of the externality and restore Pareto efficiency is to impose a tax on the consumption of $x_A$ when there is a negative externality (to "correct" A’s tendency to consume "too much" $x_A$), or subsidize $x_A$ when the externality is positive (and "correct" A’s tendency to consume "too little" $x_A$). The corrective taxes and subsidies are called "Pigouvian" after the British economist Arthur Pigou who originally proposed them as a solution to the market failure.

To compute the Pigouvian tax/subsidy in our case, suppose that there is a tax t on $x_A$. If $t < 0$, then the tax is actually a subsidy so agent A gets money for consuming x. Given t, agent A’s budget constraint becomes:

$$p_x x_A + p_y y_A = p_x \bar{x}_A + p_y \bar{y}_A - t x_A. \quad \textbf{(9)}$$

That is, a tax lowers A’s income while a subsidy increases it. Substituting for $y_A$ from the budget constraint into agent A’s objective function and differentiating with respect to $x_A$, the first order condition for agent A’s maximization problem becomes

$$4 - x_A - t = p_x. \quad \textbf{(10)}$$

Since agent B’s maximization problem remains as in the original Walrasian equilibrium without taxes/subsidies, it follows that at a Walrasian equilibrium it must be the case that,

$$4 - x_A - t = 4 - x_B. \quad \textbf{(11)}$$

Using the feasibility constraint that requires that $x_A+x_B = 4$, the condition for a Walrasian equilibrium becomes:

$$4 - x_A - t = 4 - (4 - x_A) = x_A, \Rightarrow x_A^* = \frac{4 - t}{2}. \quad \textbf{(12)}$$

Comparing equation (12) with equation (8), one can immediately see that in order to restore Pareto efficiency we need to choose t = -γ. That is, we need to tax $x_A$ at a rate of -γ when the externality is negative and subsidize $x_A$ at a rate of γ (i.e., use a negative tax γ) when the externality is positive. The reason why at the beginning we emphasized that the initial endowments of y have to be sufficiently big is that the tax/subsidy are paid in terms of units of y. Hence if agent A has to pay a tax, we need to make sure that agent A has enough units of y to make this payment. Likewise if agent A gets a subsidy, the money has to be raised from agent B so again it must be the case that agent B has enough units of y to finance the subsidy.

An alternative policy to taxing agent A when the externality is negative and subsidizing agent A when the externality is positive is to tax/subsidize agent A for consuming more or less some target level of $x_A$. Specifically, note that the market failure arises because agent A consumes 2 units of $x_A$ instead of consuming (4+γ)/2 units which is the Pareto efficient level. Now suppose that $x_A = 2$ is set as a target level of $x_A$ and now agent A has to pay a per-unit tax s for consuming more than 2 units of $x_A$ or the agent receives a per-unit subsidy s for each unit of $x_A$ below 2 units. In other words, the subsidy/tax is equal to s(2-$x_A$), where 2-$x_A$ is the deviation of agent A's from the target level. Note that if $x_A$ is below 2 the agent gets money while if it is above 2 the agent pays a tax. Given this scheme, agent A's budget constraint becomes

$$p_x x_A + p_y y_A = p_x \bar{x}_A + p_y \bar{y}_A + s(2 - x_A). \quad \textbf{(13)}$$

Therefore the first order condition for the agent's maximization problem becomes

$$4 - x_A - s = p_x. \quad \textbf{(14)}$$

Noting that equation (14) is identical to equation (10) (except that the letter s replaces the letter t), it follows that if γ < 0 (negative externality) agent A gets a subsidy for cutting its consumption of $x_A$ to 2 units, while when γ > 0 (a positive externality) agent A is taxed for failing to consume at least 2 units of $x_A$. The reason why this schemes is equivalent to the scheme we examined earlier is that the current scheme involves a transfer of 2s-s$x_A$ while the previous one involved a transfer of $tx_A$. Since 2s is a constant, it is obvious that if s = t, the two scheme will induce agent A to consume exactly the same amount and hence they are completely equivalent.

## The Coase theorem

Another way to correct the externality and restore Pareto efficiency is to assign property rights and let the two agent bargain freely with one another until they reach a Pareto efficient allocation. This solution was proposed Ronald Coase.

To see how property right might work, suppose that the law gives agent B the exclusive property rights over the use of x so that agent B can confiscate A's endowment of x and consume it all. Moreover, suppose that agent B can make agent A a take-it-or-leave offer, according to which agent B will allow agent A to consume $x_A$ units of x if in return agent A will pay agent B T dollars. If agent A rejects the offer, the two agents cannot reach any agreement (otherwise of course B's offer would not be a take-it-or-leave-it offer). In that case, agent A will consume 0 units of x while agent B will consume all 4 units.

Given this setup, agent A realizes that by rejecting B's offer his/her utility will be

$$U^A(0,\bar{y}_A) = \bar{y}_A. \qquad \textbf{(15)}$$

That is, agent A will simply consume the initial endowment of y. But, if A accepts B's offer then his/her utility is:

$$U^A(x_A,\bar{y}_A) = \left(4x_A - \frac{x_A^2}{2}\right) + \bar{y}_A - T. \qquad \textbf{(16)}$$

Comparing the two levels of utility, it is clear that the most that agent A will agree to pay agent B is:

$$T^* = 4x_A - \frac{x_A^2}{2}. \qquad \textbf{(17)}$$

Agent B anticipates T* so he/she chooses $x_A$ to maximize his/her utility subject to the resources constraint. That is, agent B solves the following problem:

$$\underset{x_A, y_A, x_B, y_B}{Max} \left(4x_B - \frac{x_B^2}{2}\right) + \gamma x_A + \bar{y}_B + T^*,$$

$$s.t. \quad x_A + x_B = 4,$$

$$\bar{y}_A + \bar{y}_B = \bar{y}. \tag{18}$$

Substituting from the feasibility constraints and for T* from equation (17) into B's objective function, B's maximization problem becomes:

$$\underset{x_A}{Max} \left(4x_A - \frac{x_A^2}{2}\right) + \left(4(4 - x_A) - \frac{(4 - x_A)^2}{2}\right) + \gamma x_A + \bar{y}_B. \tag{19}$$

This problem however is essentially the same as the one in (7) (the two differ only by a constant that does not affect the solution). Hence, B's choice of $x_A$ will be Pareto efficient. In other words, B will offer A to consume 2 units of x in return for a payment that will extract all of A's utility from doing so (relative to A's option to consume only his/her initial endowment of y).

One can now wonder if the assignment of property rights matter in this example. To examine this question, lets assume that the property rights over the use of x are now assigned to agent A. The agent can makes agent B a take-it-or-leave-it offer according to which agent B can pay agent A T dollars in return for A's agreement to allow agent B to consume units of x. Now, if agent B reject's A's offer, his/her utility will be

$$U^B(0, \bar{y}_B, x_A) = \gamma 4 + \bar{y}_B, \tag{20}$$

where the first expression on the right side of the equation is the externaility in the absence of an agreement. In that case agent A consumes all 4 units of x so the externality is $4\gamma$. If agent B accepts A's offer to exchange x's for a payment of T, then his/her utility is:

$$U^B(x_B, \bar{y}_B, x_A) = \left(4x_B - \frac{x_B^2}{2}\right) + \gamma x_A + \bar{y}_B - T. \tag{21}$$

Comparing the two utility levels, it is clear that the most that agent B will agree to pay agent A is:

$$T^* = 4x_B - \frac{x_B^2}{2} + \gamma(x_A - 4). \tag{22}$$

Anticipating T*, agent A will choose $x_A$ to maximize his utility subject to the resources constraint by solving the following problem:

$$\begin{aligned} &\underset{x_A, y_A, x_B, y_B}{Max} \left(4x_A - \frac{x_A^2}{2}\right) + \bar{y}_A + T^*, \\ &s.t. \quad x_A + x_B = 4, \\ &\qquad\;\; \bar{y}_A + \bar{y}_B = \bar{y}. \end{aligned} \tag{23}$$

Substituting from the constraints for $x_B$ and $y_B$ and substituting for T* from equation (24), B's maximization problem becomes:

$$\underset{x_A}{Max} \left(4x_A - \frac{x_A^2}{2}\right) + \left(4(4 - x_A) - \frac{(4 - x_A)^2}{2}\right) + \gamma x_A + \left(\bar{y}_B - \gamma 4\right). \tag{24}$$

This problem is equivalent to problem (19), implying that the precise allocation of property rights is immaterial in the sense that bargaining between the two agents lead to Pareto efficiency irrespective of who has the property rights.