GE in production economies

Yossi Spiegel

Consider a production economy with two agents, two inputs, K and L, and two outputs, x and y. The two agents have utility functions

$$U(x^A, y^A), \qquad U^B(x^B, y^B), \tag{1}$$

where x^A and y^A is agent A's allocation and x^B and y^B is agent B's allocation. The agents have initial endowments of inputs, $(\bar{K}^A, \bar{L}^A)$ and $(\bar{K}^B, \bar{L}^B)$. The agents sell their input endowments to a single firm that uses them to produce the outputs, x and y, according the following production functions:

$$x = f(K_x, L_x), \qquad y = g(K_y, L_y), \tag{2}$$

where K_x and L_x are the quantities of the two inputs used in the production of good x and K_y and L_y are the quantities of the two inputs used in the production of good y. The firm sells the two outputs to the two agents. The firm belongs to the two agents and pays its profits as dividends to the two agents according to their ownership shares, α^A and α^B, where $\alpha^A+\alpha^B$.

Walrasian equilibrium

Let w_K and w_L be the prices of the two inputs and let p_x and p_y be the prices of the two outputs. A Walrasian equilibrium is a vector of prices, $\Omega^* = (w_K^*, w_L^*, p_x^*, p_y^*)$ such that the markets for the two inputs and the two output clear.

To characterize the Walrasian equilibrium, consider the firm's problem. The firm wishes to maximize profits by choosing K_x, L_x, K_y, and L_y to maximize its profits. It's problem then is:

$$\underset{K_x, K_y, L_x, L_y}{Max}\ \pi = p_x f(K_x, L_x) + p_y g(K_y, L_y) - w_K(K_x + K_y) - w_L(L_x + L_y). \tag{3}$$

The first order conditions for an interior solution to the firm's problem are:

$$\frac{\partial \pi}{\partial K_x} = p_x f_K(K_x, L_x) - w_K = 0, \tag{4}$$

$$\frac{\partial \pi}{\partial L_x} = p_x f_L(K_x, L_x) - w_L = 0, \tag{5}$$

$$\frac{\partial \pi}{\partial K_y} = p_x g_K(K_y, L_y) - w_K = 0, \tag{6}$$

and

$$\frac{\partial \pi}{\partial L_y} = p_y g_L(K_y, L_y) - w_L = 0. \tag{7}$$

If we solve the 4 first order conditions we get the demands of the firm to inputs as functions of the vector of prices, $\Omega = (w_K, w_L, p_x, p_y)$ (note that we do not use asterisks at this point since the demand are defined for any vector of prices even if it does is not a Walrasian equilibrium). The demands are $K_x(\Omega)$, $L_x(\Omega)$, $K_y(\Omega)$, and $L_y(\Omega)$. The outputs of the firm are

$$x^s(\omega) = f(K_x(\Omega), L_x(\Omega)), \qquad y^s(\Omega) = g(K_y(\Omega), L_y(\Omega)). \tag{8}$$

The profit of the firm at the optimum is

$$\pi(\Omega) = p_x x^s(\Omega) + p_y y^s(\Omega) - w_K(K_x(\Omega) + K_y(\Omega)) - w_L(L_x(\Omega) + L_y(\Omega)). \tag{9}$$

Next consider the two agents. The two agents need to choose the quantities of x and y that wish to consume subject to their budget constraints. The income of the two agents comes from two sources: (i) selling the initial endowments of inputs to the firm, and (ii) dividends from the firm. Hence the income of agents is:

$$M^i(\Omega) = w_K \bar{K}^i + w_L \bar{L}^i + \alpha^i \pi(\Omega), \qquad i = A,B. \tag{10}$$

The maximization problem that agent i (i = A,B) solve is:

$$\underset{x^i,y^i}{Max}\, U^i(x^i,y^i) \quad s.t. \quad p_x x^i + p_y y^i = M^i(\Omega). \tag{11}$$

To solve this problem, let's write the associated Langragian:

$$\underset{x^i,y^i,\lambda}{Max} \ \mathfrak{L}(x^i,y^i,\lambda) = U^i(x^i,y^i) + \lambda\left(p_x x^i + p_y y^i - M^i(\Omega)\right). \tag{12}$$

The first order conditions for an interior solution to the consumer's problem are:

$$\frac{\partial \mathfrak{L}(x^i,y^i,\lambda)}{\partial x^i} = U_x^i(x^i,y^i) - \lambda p_x = 0, \tag{13}$$

$$\frac{\partial \mathfrak{L}(x^i,y^i,\lambda)}{\partial x^i} = U_y^i(x^i,y^i) - \lambda p_y = 0, \tag{14}$$

and

$$\frac{\partial \mathfrak{L}(x^i,y^i,\lambda)}{\partial \lambda} = p_x x^i + p_y\, y^i - M^i(\Omega) = 0. \tag{15}$$

Solving the 3 first order conditions we obtain the demands of agent i to the two outputs as a function of the vector of prices, $\Omega = (w_K, w_L, p_x, p_y)$. The prices of the inputs affect the demands of agent i for outputs because they affect the firm's profit which is part of the agent's income. The demands of the two agents for outputs are $x^A(\Omega)$, $y^A(\Omega)$, $x^B(\Omega)$, and $y^B(\Omega)$.

A Walrasian equilibrium is a vector of prices, $\Omega^* = (w_K^*, w_L^*, p_x^*, p_y^*)$ such that the following market clearing conditions hold:

$$K_x(\Omega) + K_y(\Omega) = \bar{K}^{A} + \bar{K}^{B}, \tag{16}$$

$$L_x(\Omega) + L_y(\Omega) = \bar{L}^{A} + \bar{L}^{B}, \tag{17}$$

$$x^{A}(\Omega) + x^{B}(\Omega) = x^{s}(\Omega), \tag{18}$$

and

$$y^{A}(\Omega) + y^{B}(\Omega) = y^{s}(\Omega). \tag{19}$$

The first two equations are the market clearing conditions in the markets of inputs K and L while the last two equations are the market clearing conditions in the markets of outputs x and y.

Walras Law

By Walras law we can ignore one of the 4 market clearing conditions and normalize the price of the associated good to 1. To see why this is true, let's sum up the budget constraints of agents A and B:

$$p_x\left(x^{A}(\Omega) + x^{B}(\Omega)\right) + p_y\left(y^{A}(\Omega) + y^{B}(\Omega)\right) = M^{A}(\Omega) + M^{B}(\Omega). \tag{20}$$

Using equation (10), this equation becomes:

$$\begin{aligned} &p_x\left(x^{A}(\Omega) + x^{B}(\Omega)\right) + p_y\left(y^{A}(\Omega) + y^{B}(\Omega)\right) \\ &= w_K\bar{K} + w_L\bar{L} + (\alpha^{A} + \alpha^{B})\,\pi(\Omega). \end{aligned} \tag{21}$$

where,

$$\bar{K} \equiv \bar{K}^{A} + \bar{K}^{B}, \qquad \bar{L} \equiv \bar{L}^{A} + \bar{L}^{B}. \tag{22}$$

But since $\alpha^A+\alpha^B = 1$ and substituting for $\pi(\Omega)$ from equation (9) we obtain:

$$p_x\left(x^A(\Omega) + x^B(\Omega)\right) + p_y\left(y^A(\Omega) + y^B(\Omega)\right)$$
$$= w_K\bar{K} + w_L\bar{L} + p_x x^s(\Omega) + p_y y^s(\Omega) \tag{23}$$
$$- w_K\left(K_x(\Omega) + K_y(\Omega)\right) - w_L\left(L_x(\Omega) + L_y(\Omega)\right).$$

Reorganizing terms, this equation becomes:

$$p_x\left(x^A(\Omega) + x^B(\Omega) - x^s(\Omega)\right) + p_y\left(y^A(\Omega) + y^B(\Omega) - y^s(\Omega)\right)$$
$$= w_K\left(\bar{K} - K_x(\Omega) - K_y(\Omega)\right) + w_L\left(\bar{L} - L_x(\Omega) - L_y(\Omega)\right). \tag{24}$$

From this equation it is clear that if 3 markets clear than the last market must clear as well. For instance, suppose the markets for x, y, and K clear. Then the left side of the equation vanishes and the first term on the right side vanishes as well. The equation then becomes:

$$w_L\left(\bar{L} - L_x(\Omega) - L_y(\Omega)\right) = 0, \tag{25}$$

which means that the market clearing condition in the market for L holds.

Walras law says that we can ignore one of the markets. Which one would that be? Given that we have a complete discretion in choosing this market, we will naturally ignore the market for which the market clearing condition is the hardest to solve and normalize the price of the good, whose market we ignore, to 1.

Pareto efficiency

To characterize the set of Pareto efficient allocations, let us begin by describing what the economy can produce. In other words, let us first find out the maximal combinations of x and y that can be produced using the available endowments of inputs. To this end, let suppose that the economy already produces a certain amount of y, say $\bar{y}$ units of y. How many units of x can the economy produce in addition to the $\bar{y}$ units of y? To answer this question we must solve the following problem:

$$\begin{aligned} &\underset{\substack{K_x, L_x \\ K_y, L_y}}{Max}\ f(K_x, L_x) \\ &s.t. \quad g(K_y, L_y) = \bar{y} \\ &\qquad K_x + K_y = \bar{K}^A + \bar{K}^B \equiv \bar{K} \\ &\qquad L_x + L_y = \bar{L}^A + \bar{L}^B \equiv \bar{L} \end{aligned} \tag{26}$$

In this problem we maximize the production of x, taking into account that some of the inputs are already used in the production of y and taking into account that we cannot use more inputs than are available in the economy.

The above problem can be simplified by substituting from the inputs constraints into the second constraint:

$$\begin{aligned} &\underset{K_x, L_x}{Max}\, f(K_x, L_x) \\ &s.t. \quad g(\bar{K} - K_x, \bar{L} - L_x) = \bar{y} \end{aligned} \tag{27}$$

To solve this problem, let's write the associated Langragian:

$$\underset{K_x, L_x, \lambda}{Max}\ \mathfrak{L}(K_x, L_x, \lambda) = f(K_x, L_x) + \lambda\left(g\left(K - K_x, \bar{L} - L_x\right) - \bar{y}\right). \tag{28}$$

The first order conditions for an interior solution to this maximization problem are:

$$\frac{\partial \mathfrak{L}(K_x, L_y, \lambda)}{\partial K_x} = f_K(K_x, L_x) - \lambda g_K\left(\bar{K} - K_x, \bar{L} - L_x\right) = 0, \tag{29}$$

$$\frac{\partial \mathfrak{L}(K_x, L_y, \lambda)}{\partial L_x} = f_L(K_x, L_x) - \lambda g_L\left(\bar{K} - K_x, \bar{L} - L_x\right) = 0, \tag{30}$$

and

$$\frac{\partial \mathfrak{L}(K_x, L_x, \lambda)}{\partial \lambda} = g\left(\bar{K} - K_x, \bar{L} - L_x\right) - \bar{y} = 0. \qquad \textbf{(31)}$$

Now, let’s divide equation (30) by equation (29). This yields:

$$-\frac{f_L(K_x, L_x)}{f_K(K_x, L_x)} = -\frac{g_L\left(\bar{K} - K_x, \bar{L} - L_x\right)}{g_K\left(\bar{K} - K_x, \bar{L} - L_x\right)}. \qquad \textbf{(32)}$$

The ratio on the left side of the equation is the technical rate of substitution in the production of x. It determines the amount of K we can give up when we use an extra unit of L in the production of x such that the total output of x will remain constant. The ratio on the right side of the equation is the technical rate of substitution in the production of y, assuming that when we produce y we use the available input that are left after we already used certain amounts of K and L to produce x. Hence, equation (32) says that in a Pareto efficient allocation, it must be the case that the technical rates of substitution in the production of x and y are the same. Put differently, the equation says that the slopes of isoquants of x and isoquants of y must be tangent.

Solving equations (31) and (32) (recall that equation (32) is a combination of equations (29) and (30)), we obtain the set of allocations of K and L at which production is efficient: there is no way to reallocate the inputs across the two production processes (the one for producing x and the other for producing y) and obtain higher quantities of both x and y. More precisely, there is no way to produce more units of x without lowering the production of y and vice versa. The combinations of K_x and L_x that satisfy equations (31) and (32) correspond to a contract curve in an "inputs" edgeworth box in which the horizontal axis is the amount of L, the vertical axis is the amount of K, the origin (the point (0,0)) is point where all inputs are used in the production of y, and $(\bar{L},\bar{K})$ is point where all inputs are used in the production of x.

Let (K_x^*, L_x^*) be the solution to equations (31) and (32). Therefore, the amounts of x and y that can be produced are given by

$$x^* = f(K_x^*, L_x^*), \qquad y^* = g(\bar{K} - K_x^*, \bar{L} - L_x^*). \qquad \textbf{(33)}$$

Equation (32) describes some relationship between K and L that must hold on the contract curve in the inputs edgeworth box. If the marginal products of the two inputs are positive in the production of both x and y then the contract curve will necessarily start and the origin, end at $(\bar{L},\bar{K})$, and will be upward sloping. This implies in turn that equation (32) which describes the contract curve defines a 1:1 relationship between K and L: for each K there will be a unique L on the contract curve and for each L there will be a unique K on the contract curve.

Using the fact that by equation (32) there is a 1:1 relationship between K and L, equation (32) defines x* and y* in terms of a single variable, since if we fix L_x* then K_x* is determined by equation (32) and if we fix K_x* then L_x* is determined by equation (32). Since both x* and y* depend on a single variable, say L_x*, we can isolate L_x* from the equation x* = f*(K_x*,L_x*) and substitute it into the equation y* = g*($\bar{K}$-K_x*,$\bar{L}$-L_x*). This will give us a relationship between y* and x*. This relationship is called the **Production Possibilities Frontier**, or **PPF** for short. The PPF is the analog of the Utilities Possibility Frontier (UPF) in the case of an exchange economy. The PPF describes the trade-off between y and x as we move along the contract curve in the inputs edgeworth box. In other words, the PPF tells us what are the maximal combinations of x and y that the economy can produce when the inputs are used efficiently. In general, the PPF can be described by a function,

$$T(x^*,y^*) = 0. \tag{34}$$

As mentioned above, this function is obtained by using equation (32) to determine K_x* in terms of L_x*, then isolating L_x* from the equation x* = f*(K_x*,L_x*) and substituting it into the equation y* = g*($\bar{K}$-K_x*,$\bar{L}$-L_x*), thereby expressing y* in terms of x*.

To illustrate how we find the PPF, consider the case where

$$f(K_x,L_x) = K_x^{\alpha} L_x^{\beta}, \qquad g(K_y,L_y) = K_y^{\alpha} L_y^{\beta}. \tag{35}$$

Then,

$$f_L(K_x,L_x) = \beta K_x^{\alpha} L_x^{\beta-1}, \qquad f_K(K_x,L_x) = \alpha K_x^{\alpha-1} L_x^{\beta}. \tag{36}$$

and

$$g_L(K_y,L_y) = \beta K_y^{\alpha} L_y^{\beta-1}, \qquad g_K(K_y,L_y) = \alpha K_y^{\alpha-1} L_y^{\beta}. \tag{37}$$

Therefore, equation (32) can be written in this case as

$$-\frac{\beta K_x}{\alpha L_x} = -\frac{\beta\left(\bar{K} - K_x\right)}{\alpha\left(\bar{L} - L_x\right)}. \qquad \textbf{(38)}$$

Solving this equation for K_x yields,

$$K_x^* = \frac{\bar{K}}{\bar{L}} L_x^*. \qquad \textbf{(39)}$$

That is, the contract curve in the production edgeworth box is simply the diagonal of the box, i.e., the straight line that connects the origin with the point $(\bar{L},\bar{K})$. Substituting in equation (35) and recalling that $K_y^* = \bar{K}\text{-}K_x^*$ and $L_y^* = \bar{L}\text{-}L_x^*$, we obtain

$$\begin{aligned} x^* &= \left(\frac{\bar{K}}{\bar{L}} L_x^*\right)^{\alpha} L_x^{*\beta} = \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} (L_x^*)^{\alpha+\beta}, \\ y^* &= \left(\bar{K} - \frac{\bar{K}}{\bar{L}} L_x^*\right)^{\alpha} \left(\bar{L} - L_x^*\right)^{\beta} = \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \left(\bar{L} - L_x^*\right)^{\alpha+\beta}. \end{aligned} \qquad \textbf{(40)}$$

Using the expression for x* we get:

$$L_x^* = (x^*)^{\frac{1}{\alpha+\beta}} \left(\frac{\bar{L}}{\bar{K}}\right)^{\frac{\alpha}{\alpha+\beta}}. \qquad \textbf{(41)}$$

We can now substitute for L_x^* into the expression for y* and obtain:

$$y^* = \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \left(\bar{L} - (x^*)^{\frac{1}{\alpha+\beta}} \left(\frac{\bar{L}}{\bar{K}}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\alpha+\beta} = \left(\bar{K}^{\frac{\alpha}{\alpha+\beta}} \bar{L}^{1-\frac{\alpha}{\alpha+\beta}} - (x^*)^{\frac{1}{\alpha+\beta}}\right)^{\alpha+\beta}. \qquad \textbf{(42)}$$

The PPF is therefore given by the expression

$$T(x^*,y^*) = y^* - \left(\bar{K}^{\frac{\alpha}{\alpha+\beta}}\bar{L}^{1-\frac{\alpha}{\alpha+\beta}} - (x^*)^{\frac{1}{\alpha+\beta}}\right)^{\alpha+\beta} = 0. \quad \textbf{(43)}$$

As you can see, this expression is pretty complex. However when $\alpha+\beta = 1$ (in that case the production of both x and y exhibits constant return to scale), the expression can be simplified substantially:

$$T(x^*,y^*) = y^* - \bar{K}^{\alpha}\bar{L}^{1-\alpha} + x^* = 0. \quad \textbf{(44)}$$

In other words, the PPF is given by the linear expression

$$y^* = \bar{K}^{\alpha}\bar{L}^{1-\alpha} - x^*. \quad \textbf{(45)}$$

Having found the PPF, we are now ready to describe the set of Pareto efficient allocations. This set is determined by the problem:

$$\begin{aligned} &\underset{\substack{x^A,y^A \\ x^B,y^B}}{Max}\ U^A(x^A,y^A) \\ &s.t.\quad U^B(x^B,y^B) = \bar{U}^B \\ &\qquad x^A + x^B = \bar{x} \\ &\qquad y^A + y^B = \bar{y} \\ &\qquad T(\bar{x},\bar{y}) = 0 \end{aligned} \quad \textbf{(46)}$$

We can simply matters by rewriting this problem as follows:

$$\begin{aligned} &\underset{\substack{x^A,y^A \\ x^B,y^B}}{Max}\ U^A(x^A,y^A) \\ &s.t.\quad U^B(x^B,y^B) = \bar{U}^B \\ &\qquad T(x^A + x^B, y^A + y^B) = 0 \end{aligned} \quad \textbf{(47)}$$

To solve this problem, let’s write the associated Langragian:

$$\underset{x^{A},y^{A},x^{B},y^{B},\lambda,\mu}{Max} \quad \mathcal{L}(x^{A},y^{A},x^{B},y^{B},\lambda,\mu) = U^{A}(x^{A},y^{A}) + \lambda\left(U^{B}(x^{B},y^{B}) - \bar{U}^{B}\right) + \mu\left(T(x^{A}+x^{B},y^{A}+y^{B}) - 0\right). \qquad \textbf{(48)}$$

The first order conditions for an interior solution to this maximization problem are:

$$\frac{\partial \mathcal{L}}{\partial x^{A}} = U_{x}^{A}(x^{A},y^{A}) - \mu\, T_{x}(x^{A}+x^{B},y^{A}+y^{B}) = 0, \qquad \textbf{(49)}$$

$$\frac{\partial \mathcal{L}}{\partial y^{A}} = U_{y}^{A}(x^{A},y^{A}) - \mu\, T_{y}(x^{A}+x^{B},y^{A}+y^{B}) = 0, \qquad \textbf{(50)}$$

$$\frac{\partial \mathcal{L}}{\partial x^{B}} = \lambda\, U_{x}^{B}(x^{A},y^{A}) - \mu\, T_{x}(x^{A}+x^{B},y^{A}+y^{B}) = 0, \qquad \textbf{(51)}$$

$$\frac{\partial \mathcal{L}}{\partial y^{B}} = U_{y}^{B}(x^{A},y^{A}) - \mu\, T_{y}(x^{A}+x^{B},y^{A}+y^{B}) = 0, \qquad \textbf{(52)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = U^{B}(x^{B},y^{B}) - \bar{U}^{B} = 0, \qquad \textbf{(53)}$$

and

$$\frac{\partial \mathcal{L}}{\partial \mu} = T(x^{A}+x^{B},y^{A}+y^{B}) - 0 = 0. \qquad \textbf{(54)}$$

Equations (49)-(54) give us 6 equations in 6 unknowns. Once we solve this 6 equations we characterize the set of Pareto efficient allocations for our production economy. However, even without solving this system of equations we can say quite a bit about the properties of the Pareto efficient allocations. To see that, notice that if we divide equation (49) by equation (50) then we obtain:

$$-\frac{U_x^A(x^A, y^A)}{U_y^A(x^A, x^B)} = -\frac{T_x(x^A + x^B, y^A + y^B)}{T_y(x^A + x^B, y^A + y^B)}. \qquad \textbf{(55)}$$

Likewise, if we divide equation (51) by equation (52) then we obtain:

$$-\frac{U_x^B(x^A, y^A)}{U_y^B(x^A, x^B)} = -\frac{T_x(x^A + x^B, y^A + y^B)}{T_y(x^A + x^B, y^A + y^B)}. \qquad \textbf{(56)}$$

The expressions on the left side of equations (55) and (56) are the marginal rates of substitution of the two agents, or equivalently, the slopes of their indifference curves. The expression on the right hand side of both equations is the slope of the PPF. This slope determines the amount of y that the economy needs to forgo if it wishes to increase the production of x by one unit, given that production is done efficiently.

Taken together, equations (55) and (56) say that the marginal rates of substitution for the two agents must be equal to one another and moreover equal to the slope of the PPF:

$$MRS^A = MRS^B = -\frac{T_x(x^A + x^B, y^A + y^B)}{T_y(x^A + x^B, y^A + y^B)}. \qquad \textbf{(57)}$$

That is, the slopes of the indifference curves of A and B must be equal, implying that their indifference curves are tangent at a Pareto efficient allocation.

The first welfare Theorem: The Walrasian equilibrium is Pareto efficient

If we take equations (13) and (14) (both of which are satisfied at the Walrasian equilibrium), and divide them by one another we get:

$$MRS^i \equiv \frac{U_x^i(x^i, y^i)}{U_y^i(x^i, y^i)} = \frac{p_x}{p_y}, \qquad i = A, B. \qquad \textbf{(58)}$$

Now we can also write the firm’s problem as:

$$\underset{x,y}{Max}\ \pi(x,y) = p_x x + p_y y - C(x,y), \tag{59}$$

where C(x,y) is the cost function of the firm that tells us what is the least costly way to produce the combination (x,y). The first order conditions for the firm's problem are:

$$\frac{\partial \pi(x,y)}{\partial x} = p_x - C_x(x,y) = 0, \tag{60}$$

and

$$\frac{\partial \pi(x,y)}{\partial y} = p_y - C_y(x,y) = 0. \tag{61}$$

If we divide equations (60) and (61) by one another we get that in a Walrasian equilibrium it must be the case that:

$$\frac{C_x(x,y)}{C_y(x,y)} = \frac{p_x}{p_y}. \tag{62}$$

Now, note that on the PPF, the expenditure of the firm on inputs must be constant since the firm buys the entire endowments of the inputs no matter where on the PPF we are (if the firm does not buy all the inputs then the economy can produce more of x or more of y or both so the allocation is inefficient). Hence, on the PPF, the following equation must hold:

$$C(x,y) = w_K \bar{K} + w_L \bar{L}. \tag{63}$$

In other words, the combinations of x and y that are on the PPF must satisfy equation (63).

Given the above equation we can now ask what is the slope of the PPF. That is, we can ask the following question: "How many units of y we must give up when we wish to obtain an extra unit of x such that the total expenditure on inputs will remain the same?" To answer the question, we can differentiate equation (63) with respect to both x and y. Doing that yields:

$$C_x(x,y)dx + C_y(x,y)dy = 0, \Rightarrow \left.\frac{dy}{dx}\right|_{PPF} = -\frac{C_x(x,y)}{C_y(x,y)}. \tag{64}$$

At the same time, the PPF is also described by the equation

$$T(x,y) = 0. \tag{65}$$

We can now determine the slope of the PPF by differentiating T(x,y) = 0 with respect to both x and y. This gives an answer to the question, "How many units of y we must give up when we wish to obtain an extra unit of x given that inputs are used efficiently?" Doing that yields:

$$T_x(x,y)dx + T_y(x,y)dy = 0, \Rightarrow \left.\frac{dy}{dx}\right|_{PPF} = -\frac{T_x(x,y)}{T_y(x,y)}. \tag{66}$$

Taken together, equation (64) and (66) imply that

$$\left.\frac{dy}{dx}\right|_{PPF} = -\frac{C_x(x,y)}{C_y(x,y)} = -\frac{T_x(x,y)}{T_y(x,y)} \tag{67}$$

Now recall that equation (62) shows that

$$-\frac{C_x(x,y)}{C_y(x,y)} = -\frac{p_x}{p_y}. \tag{68}$$

Together with equation (58), equations (67) and (68) show that in a Walrasian equilibrium,

$$MRS^A = MRS^B = -\frac{p_x}{p_y} = -\frac{C_x(x,y)}{C_y(x,y)} = -\frac{T_x(x,y)}{T_y(x,y)}. \tag{69}$$

Since $x = x^A + x^B$ and $y = y^A + y^B$, equation (69) implies that

$$MRS^A = MRS^B = -\frac{T_x(x^A + x^B, y^A + y^B)}{T_y(x^A + x^B, y^A + y^B)}. \tag{70}$$

This is exactly the condition for Pareto efficiency. Hence the Walrasian equilibrium is Pareto efficient.