Clarke-Groves mechanisms for optimal provision of public goods

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Consider an economy with one public good G, and one private good y. If the preferences of all agents in the economy are common knowledge, then it is fairly easy to achieve Pareto efficient provision of the public good. For instance, the agents can delegate authority to one of them (e.g., this agent plays the role of a government) and this agent then needs to decide on the size of public good and how it will be financed. If this agent is benevolent, he can achieve a Pareto efficiency by applying the Samuelson rule to determine the size of the public good and can finance it with taxes that will leave all agents better off relative to their initial situation.

To illustrate this, suppose that there are only 2 agents in the economy, A and B. The two agents have initial endowments of the private good, $\bar{y}_A$ and $\bar{y}_B$. The private good can either be consumed directly, or it can be converted to a public good. Suppose that C(G) units of the private good are needed to produce G units of the public good, where $C'(G) > 0$ and $C''(G) \leq 0$. That is, the marginal cost of producing the public good (in terms of how many units of the private good you need to give up) is increasing at a (weakly) decreasing rate. The preferences of the two agents are given by the following quasi-linear utility function:

$$U(G,y_i) = V(G,\theta_i) + y_i, \qquad i = A,B, \tag{1}$$

where V(.) is the subutility of agent i from the consumption of the public good and $\theta_i > 0$ is a (positive) taste parameter. We shall assume that V(.) is an increasing and concave function of G (i.e., $V_G(.) > 0$ and $V_{GG}(.) \leq 0$), and that V_G increases with θ_i. To understand the meaning of the last assumption, note that V_G is the marginal willingness to pay for the public good. Since the utilities of the agents are quasi-linear, V_G can be also interpreted as the number of units of the private good that an agent is willing to give up in order to get an extra unit of the public good. In other words, V_G is the marginal rate of substitution (MRS) between the public and private goods. The assumption that V_G increases with θ_i means that a person with a higher θ_i

has a higher MRS and hence is more eager to substitute private for public goods. For instance, the function $V(G,\theta_i)$ can take the form $\theta_i Ln(G)$, or $\theta_i G$, or $\theta_i G - G^2/2$. In these examples, V_G is given by θ_i/G, θ_i, and $\theta_i - G$, respectively so obviously, it is increasing with θ_i in all 3 examples.

Now suppose the agents can nominate one of them to decide on the size of the public good and how it will be financed (i.e., how many units of the private good each agent will contribute towards the production of the public good). If the nominated agent is benevolent he will look for a Pareto efficient allocation and will therefore choose G by solving the following problem:

$$\begin{aligned} \max_{G,y_A,y_B} \quad & V(G,\theta_A) + y_A \\ s.t. \quad & V(G,\theta_B) + y_B = \bar{U}^B \\ & C(G) + y_A + y_B = \bar{y}_A + \bar{y}_B. \end{aligned} \tag{2}$$

If we substitute for y_B from the second constraint into the second, then the problem becomes:

$$\begin{aligned} \max_{G,y_A} \quad & V(G,\theta_A) + y_A \\ s.t. \quad & V(G,\theta_B) + \bar{y}_A + \bar{y}_B - C(G) - y_A = \bar{U}^B. \end{aligned} \tag{3}$$

Now substituting for y_A from the constraint into the objective function we get:

$$\max_{G} \quad V(G,\theta_A) + V(G,\theta_B) - C(G) + \bar{y}_A + \bar{y}_B - \bar{U}^B. \tag{4}$$

That is, the nominated agent needs to choose G to maximize the sum of the subutilities from the public good minus the cost of the public good which is C(G) (the last 3 terms are constants that do not affect the value of G).

The first order condition for the problem is:

$$V_G(G,\theta_A) + V_G(G,\theta_B) = C'(G). \tag{5}$$

This condition is the Samuelson condition as it says that the sum of the marginal rates of substitutions (here this is also the sum of the marginal willingness to pay for the public good) should equal to marginal cost of providing the public good. The second order condition for the problem is:

$$V_{GG}(G,\theta_A) + V_{GG}(G,\theta_B) - C''(G) \leq 0, \tag{6}$$

where the inequality follows from the assumptions about V(.) and C(.). Hence, the second order condition is satisfied implying that the first order condition is sufficient for a maximum. If we assume in addition that $V_G(0,\theta_A)+V_G(0,\theta_B) > C'(0)$, then the optimal size of the public good, G*, will be strictly positive since at G = 0, the marginal benefit from the public good exceeds the associated marginal cost. This will be the case for instance if C'(0) = 0 which happens if C(G) = G or $C(G) = G^2$.

Now the (benevolent) nominated agent can determine that G* unit shall be provided. The only remaining question is how to finance it. Given G*, C(G*) units of the private good are needed to produce G*. Let g_A and g_B be the contributions of agents A and B to the production of the public good. To fully finance the public good, it must be that $g_A + g_B = C(g^*)$. Given G*, g_A and g_B, the resulting utilities of two agents are:

$$V(G^*,\theta_i) + \bar{y}_i - g_i, \qquad i = A,B, \tag{7}$$

where $\bar{y}_i - g_i$ is the amount of the private good that the agent has left after contributing g_i towards the provision of the public good. The nominated agent then can find g_A and g_B such that the utility of each agent would exceed the utility without the public good which is $\bar{y}_i$ (such g_A and g_B can be found since G* > 0 so the agents are always better-off if some public good is provided than if none so they are willing to contribute towards providing it.

To illustrate with a specific example, suppose that $V(G,\theta_i) = \theta_i G - G^2/2$ and C(G) = G and assume in addition that $\theta_A+\theta_B > 1$. Then the Samuelson condition is given by:

$$\theta_A + \theta_B - 2G = 1. \tag{8}$$

The left side of the equation is the sum of the marginal willingness to pay of the two agents for the public good and the right side is the marginal cost of providing the public good. Equation (8) implies that

$$G^* = \frac{\theta_A + \theta_B - 1}{2}. \qquad \textbf{(9)}$$

Since by assumption, $\theta_A+\theta_B > 1$, $G^* > 0$. Given G^*, the utility of agent i when this agent pays g_i for the public good, is given by:

$$\begin{aligned} &\theta_i G^* - \frac{(G^*)^2}{2} + \bar{y}_i - g_i, \\ &= \frac{(3\theta_i - \theta_j + 1)(\theta_i + \theta_j - 1)}{8} + \bar{y}_i - g_i. \end{aligned} \qquad \textbf{(10)}$$

Since the utility of agent i without a public good is $\bar{y}_i$, any g_A and g_B such that

$$\frac{(3\theta_i - \theta_j + 1)(\theta_i + \theta_j - 1)}{8} + \bar{y}_i - g_i > \bar{y}_i, \qquad \textbf{(11)}$$

will make both agents better-off relative to the case where there is no public good.

The problem however is what to do if θ_A and θ_B are not common knowledge. That is the question is how to determine the size of the public good when each agent knows his own θ_i but does not know θ_j, which is the taste parameter of the other agent. Then we cannot write the Samuelson condition (we simply do not know the parameters θ_A and θ_B) and cannot decide what G* will be. Even if one agent is nominated to decide the size of the public good, then this agent only knows his own taste parameter but not the taste parameter of the other agent. Of course, the nominated agent can always decide on some level of a public good, but suppose this agent is benevolent and wants to do the "right" thing and choose the optimal level, G*. How can he do that without knowing the other agent’s taste parameter?

An obvious way to solve the problem is to ask the two agents for their taste parameters. If the two agents are honest, then we can write the Samuelson condition as before and find G*.

But what if the agents are not honest? Then there is a question of how to induce them to reveal their taste parameters truthfully in order to choose G*. We begin by demonstrating that in general the agents will have an incentive to report a different taste parameter than they really have in order to lower the amount they have to pay for the provision of the public good. To show this, suppose that the rule is that the cost of the public good, C(G), is divided equally between the two agents. The agents are then asked to report their taste parameters, θ_A and θ_B; given these reports, equation (5) is used to determine the size of the public good and then each agents pays half of the cost. This rule is arbitrary but at least it sounds fair. We'll see that in general this rule does not work in that the agents misreport their types so the resulting size of the public good is not G*.

To this end, let $\hat{\theta}_A$ and $\hat{\theta}_B$ be the reports of the two agents and $\hat{G}$ be the size of the public good that we get by substituting $\hat{\theta}_A$ and $\hat{\theta}_B$ into equation (5) and solving for G. Then, given the rule that the cost of the public good is divided equally between the two agents, the utilities of the two agents are:

$$U(\hat{\theta}_i, \theta_i) \equiv V(\hat{G}, \theta_i) + \bar{y}_i - \frac{C(\hat{G})}{2}, \qquad i = A, B. \tag{12}$$

Will agent i report his taste parameter truthfully? To answer this question, let's differentiate the utility of agent i with respect to his report, $\hat{\theta}_i$:

$$\frac{\partial U(\hat{\theta}_i, \theta_i)}{\partial \hat{\theta}_i} = \left[V_G(\hat{G}, \theta_i) - \frac{C'(\hat{G})}{2} \right] \frac{\partial \hat{G}}{\partial \hat{\theta}_i}, \qquad i = A, B. \tag{13}$$

To understand this condition, note that the expression in the square brackets is the change in i's utility when the size of the public good increases slightly given that the agent pays half of the cost of providing the public good, and $\partial\hat{G}/\partial\hat{\theta}_i$ is the change in the size of the public good when agent i changes his report slightly. The optimal report for agent i is determined at the point where the above derivative is equal to 0. i's report is truthful, only if the optimal report of i coincides with the true taste parameter of i.

To check whether i's report will be truthful, suppose the other agent, agent j, makes a truthful report so that $\hat{\theta}_j = \theta_j$. Then, it follows from the Samuelson rule that

$$V_G(\hat{G},\hat{\theta}_A) + V_G(\hat{G},\theta_B) = C'(\hat{G}). \qquad \textbf{(14)}$$

Substituting for $C(\hat{G})$ in equation (13) we get:

$$\frac{\partial U(\hat{\theta}_i,\theta_i)}{\partial \hat{\theta}_i} = \left[\frac{2V_G(\hat{G},\theta_i) - V_G(\hat{G},\hat{\theta}_i) - V_G(\hat{G},\theta_j)}{2}\right]\frac{\partial \hat{G}}{\partial \hat{\theta}_i}, \quad i = A,B. \qquad \textbf{(15)}$$

In general, this derivative will not be vanish at $\theta_i = \hat{\theta}_j$ so agent i will not have an incentive to make a truthful report. To see this, note that if we substitute for $\theta_i = \hat{\theta}_j$ in the derivative we get:

$$\frac{\partial U(\theta_i,\theta_i)}{\partial \hat{\theta}_i} = \left[\frac{V_G(\hat{G},\theta_i) - V_G(\hat{G},\theta_j)}{2}\right]\frac{\partial \hat{G}}{\partial \hat{\theta}_i}, \quad i = A,B. \qquad \textbf{(16)}$$

Since we assumed that V_G is increasing with θ_i, this derivative vanishes only in the special case where $\theta_i = \theta_j$, in which case, the two agents happen to have the exact same taste parameters. Otherwise, the derivative does vanish when $\theta_i = \hat{\theta}_j$ so agent i will not have an incentive to misreport his taste parameter.

So far we only saw that an equal division rule does not work in general, so perhaps there are other rules that will work. However it turns out that many other rules (e.g., proportional division according to the relative sizes of $\hat{\theta}_A$ and $\hat{\theta}_B$) will not work as well. So how do we get around this problem and induce the agents to report their types truthfully? An answer to this question was provided independently by Ed Clarke (*Public Choice*, 1971) and Ted Groves (Unpublished manuscript, 1969, and *Econometrica*, 1973). In what follows we shall refer to this proposal as the Clarke-Groves mechanism.[1] Their proposed solution is as follows:

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[1] For a nice introduction to Clarke-Groves mechanisms see Ed Clarke's webpage at http://clarke.pair.com.

1. Ask each agent to report his type.
2. Substitute the reported types, $\hat{\theta}_A$ and $\hat{\theta}_B$, in the Samuelson condition (equation (5)) to determine the size of the public good.
3. Let $\hat{G}$ be the size of the public good given the reports $\hat{\theta}_A$ and $\hat{\theta}_B$. Charge each agent the amount $g_i = C(\hat{G})-V(\hat{G},\hat{\theta}_j)$ which is the difference between the cost of providing the public good and the reported subutility of the other agent from the public good.

To see how the Clarke-Groves mechanism works, note that given the reports, the size of the public good is chosen optimally. Hence if the two agents make truthful reports then the size of the public good will be optimal. The remaining question then is will agent i make a truthful report? To answer this question, let's substitute g_i into i's utility function. Then the utility of i is:

$$U(\hat{\theta}_i,\theta_i) \equiv V(\hat{G},\theta_i) + V(\hat{G},\hat{\theta}_j) - \frac{C(\hat{G})}{2} + \bar{y}_i, \qquad i = A,B. \tag{17}$$

Now if agent i make a truthful report, then equation (17) becomes equivalent to equation (4) which is the problem of the benevolent agent who need to decide on the size of the public good. Clearly, agent i will have an incentive to report his type truthfully since his objective now coincides with that of the benevolent agent. Indeed, if we differentiate the utility of agent i with respect to his report, $\hat{\theta}_i$ we get:

$$\frac{\partial U(\hat{\theta}_i,\theta_i)}{\partial \hat{\theta}_i} = \left[V_G(\hat{G},\theta_i) + V_G(\hat{G},\hat{\theta}_j) - C'(\hat{G})\right]\frac{\partial \hat{G}}{\partial \hat{\theta}_i}, \qquad i = A,B. \tag{18}$$

By the Samuelson condition, the square bracketed expression vanishes when $\hat{\theta}_i = \theta_i$ (note that the Samuelson condition is solved with $\hat{\theta}_i$ and $\hat{\theta}_j$ since θ_i and θ_j are not known; what is known is only what i and j report which is $\hat{\theta}_i$ and $\hat{\theta}_j$). Hence, equation (18) says that agent i has an incentive to make a truthful report. Since this is true no matter what agent j reports, we can say in the language of game theory making that a truthful report is a dominant strategy for i because a truthful report is optimal irrespective of what agent j does. Since both agents make truthful reports, the resulting size of the public good is the Pareto efficient size, G*.

To illustrate with an example, suppose again that $V(G,\theta_i) = \theta_i G - G^2/2$ and C(G) = G. Then from the Samuelson condition it follows that:

$$\hat{G} = \frac{\hat{\theta}_A + \hat{\theta}_B - 1}{2}. \quad \textbf{(19)}$$

The payment that agent i pays is

$$g_i = \hat{G} - \left(\hat{\theta}_j \hat{G} - \frac{G^{\hat{2}}}{2}\right). \quad \textbf{(20)}$$

That is, agent i pays the total cost of the public good which is equal to Ĝ in this example, minus the reported subutility of agent j (note that since we do not know the actual subutility of j we use agent j's reported subutility to compute the payment that i will make). The resulting utility of agent i according to the Clarke-Groves mechanism is:

$$\begin{aligned} U(\hat{\theta}_i, \theta_i) &= \left(\theta_i \hat{G} - \frac{\hat{G}^2}{2}\right) + \bar{y}_i - \left(\hat{G} - \left(\hat{\theta}_j \hat{G} - \frac{G^{\hat{2}}}{2}\right)\right) \\ &= (\theta_i + \hat{\theta}_j)\hat{G} - \hat{G}^2 - \hat{G} + \bar{y}_i. \end{aligned} \quad \textbf{(21)}$$

Differentiating this expression with respect to i's report we get:

$$\frac{\partial U(\hat{\theta}_i, \theta_i)}{\partial \hat{\theta}_i} = \left(\theta_i + \hat{\theta}_j - 2\hat{G} - 1\right) \frac{\partial \hat{G}}{\partial \hat{\theta}_i}. \quad \textbf{(22)}$$

Substituting for $\hat{G}$ from equation (19) we get:

$$\frac{\partial U(\hat{\theta}_i,\theta_i)}{\partial \hat{\theta}_i} = \left(\theta_i + \hat{\theta}_j - (\hat{\theta}_i + \hat{\theta}_j - 1) - 1\right)\frac{\partial \hat{G}}{\partial \hat{\theta}_i} \qquad \textbf{(23)}$$

$$= \left(\hat{\theta}_i - \theta_i\right)\frac{\partial \hat{G}}{\partial \hat{\theta}_i}.$$

From this condition it is obvious that the optimal report is truthful since then the derivative vanishes to what we have found is the optimal report. Moreover this truthful report is optimal irrespective of what j reports so a truthful report is a dominant strategy for i.

There is one big problem however with the Clarke-Groves mechanism. The problem is that given the mechanism, the combined contributions of the two agents towards the public good are:

$$g_A + g_B = 2C(G^*) - V(G^*,\theta_A) - V(G^*,\theta_B). \qquad \textbf{(24)}$$

This is because the size of the public good is G* and each agent pays the cost C(G*) minus the subutility of the other agents. From equation (24) it follows that the combined contributions of the two agents will be equal to the cost of providing the public good only in the special case where $V(G^*,\theta_A)+V(G^*,\theta_B) = C(G^*)$, i.e., only if the benefit from the public good is exactly equal to the cost of providing it. In general however this is not the case so the mechanism will either generate a surplus if the contributions exceed C(G*) or a deficit if the contributions fall short of C(G*).

In the context of the example we saw above, the combined contibutions of the two agents are:

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\begin{aligned}
g_A + g_B &= 2G^* - (\theta_A + \theta_B)G^* - 2\frac{(G^*)^2}{2} \\
&= (2 - \theta_A - \theta_B - G^*)G^* \qquad \textbf{(25)} \\
&= \frac{(5 - 3\theta_A - 3\theta_B)(\theta_A + \theta_B - 1)}{2}.
\end{aligned}
$$

This expression is obviously not going to be equal to 0 unless $\theta_A+\theta_B = 5/3$ (recall that by assumption, $\theta_A+\theta_B > 1$, otherwise $G^* = 0$). So unless in a knife-edge case, the mechanism in this example will in general either create a deficit if $\theta_A+\theta_B < 5/3$ or a surplus if $\theta_A+\theta_B > 5/3$.