1. Consider the following one-period ultimatum game. The gamemaster makes available $1 (100 pennies) to split between player 1 and player 2. Player 1 moves first and proposes a split of the $1 – (x, 100-x) – where x is the number of pennies that player 1 will keep, and 100-x is the number of pennies that player 2 will keep. Player 1's proposal must be divisible into pennies (so player 1 has only a finite number of actions which he can take). After hearing the proposal, player 2 decides either to accept or reject it. If he accepts the proposal, the players split the $1 according to the proposal. If he rejects the proposal, the gamemaster takes back the $1 and the players get nothing.

a) What are player 1's possible strategies? What are player 2's possible strategies (there are so many you won't be able to list them all – just describe them)? Remember that in sequential move games like this one, there is an important difference between a player's actions and strategies. Sketch the game tree.
b) There are 2 backward induction solutions (i.e. subgame perfect Nash equilibria) to this game. Find them.
c) Find a Nash equilibrium to this game which is not a backward induction solution (i.e. is not subgame perfect). Hint: there are some Nash equilibria where player 2 walks away very happy. Hint hint: there are lots of these in fact.
d) Now suppose the game is extended into a two-period ultimatum game as follows. If the first period ends with player 2 rejecting player 1's offer, the game ends there – there is no second period extension. But, if the first period ends with player 2 accepting, then the second period is enacted. In the second period, player 2 moves first and makes a proposal on how to split the $1 (divisible by pennies again). After hearing the proposal, player 1 now decides to either accept or reject it (so the roles are reversed). The second stage payoffs depend on the response to the proposal in the same way they did in period one. What are the backward induction solutions to this two-period ultimatum game?

2. Agent 1's utility function is $U_1=X_1*Y_1$ and his initial endowments are $X_1=6$ and $Y_1=3$. Agent 2's utility function is $U_2=Y_2+2*X_2$ and his initial endowments are $X_2=4$ and $Y_2=7$.

a) Draw an Edgeworth box showing the initial endowment point and sketching few indifference curves;
b) Solve for the competitive equilibrium following these guidelines:
   i) Make good Y the numeraire, i.e. set its price to be equal to 1;
   ii) Calculate each agent's demand for good X as a function of p_x;
   iii) Equate sum of demands from ii) to the sum of initial endowments of X and solve for p_x;
   iv) Use demands calculated in ii) and p_x calculated in iii) to solve for quantity of good X consumed by each customer in competitive equilibrium;
   v) Using budget constraints to calculate the remaining quantities;
c) Graph the competitive equilibrium point in the Edgeworth box;

3. Firm A could produce butter at the rate of 6 pounds per hour and milk at the rate 10 bottles per hour. Firm B could produce butter at the rate 4 ponds per hour and milk at the rate 8 bottles per hour. Both firms are producing for 8 hours during the working day.

a) What is the maximum amount of milk which both of the firms could produce together during a working day;
b) What is the maximum amount of butter which both of the firms could produce together during a working day;

c) Which firm has a competitive advantage in producing milk;
d) At the point where both firms are producing only the good they have comparative advantage in, how much milk and how much butter is produced;
e) Draw the joint PPF of these firms and calculate its slopes.

4. Consider a production economy where Fez is the only consumer. Fez has preferences over beer (E) and brownies (R) which can be represented by: $U_F = E + 3*R$. He is endowed with 10 bottles of beer and no brownies. In this economy, there is a magic oven which can turn beer into brownies according to: $R = e^{2/3}$, where e is the input of beer. Fez owns the only beer in the economy which can be used as input. Fez owns the company which owns the magic oven, but because of language barriers and immigration laws, instead of operating the oven outright, Fez and the company (which owns the magic oven) interact in competitive markets. Brownies are traded at a given price p^R, and beer is traded at a given price p^E.

a) Write down the company's profit maximization problem with the price of beer normalized to one ($p^E = 1$). Using the production function, derive the equation characterizing p^R as a function of R.
b) Using the relationship between p^R and R, write down an expression for profits as a function of R.
c) Write down Fez's budget constraint. Since he is the owner of the company which owns the magic oven, Fez gets the company's profits as part of his income. Using the budget equation, solve Fez's consumer maximization problem.
d) Using the first order conditions and other equations you have derived, find the competitive equilibrium values for p^R, R, E, and e.

5. Oscar and Maria are neighbors whose backyards are currently vacant and need to be developed. Oscar's backyard is 10 hectares in size, and so is Maria's. There are two types of property developers that do business in this world: lawn developers and garden developers. The process of landscaping a backyard in this world is a little strange: in order to turn 3 hectares of vacant land into, say, a lawn, the owner must first sell the 3 hectares to the developer. The developer takes the land, transforms it into lawnspace, and then sells the new good back to the person. (So these lawn and garden producers are more like 'real-estate developers' – who buy land and sell finished property – than 'landscapers' – who are hired by land owners.)

The lawn developer produces a private good; that is, she turns vacant land into a lawn which only the buyer can consume and enjoy (e.g. Oscar can't cross into Maria's property to play football on her lawn). The garden developer produces a public good; that is, he turns land into a garden which both neighbors can enjoy (e.g. Oscar can enjoy the view of Maria's garden from his property while Maria is doing the same).

The lawn developer can turn land into lawn on a one-to-one basis, so the lawn production function is $L = e^L$, where the output L is hectares of lawn produced, and the input e^L is hectares of land bought from Oscar and Maria (so $e^L = e^L{}_O + e^L{}_M$). The garden production function is trickier: $G = (e^G)^{0.8}$ where G is output of garden and e^G is the total input of land bought for that use ($e^G = e^G{}_O + e^G{}_M$). Notice that if all 20 units of backyard were turned into garden, the output would be less than 20 (say, due to space needed for trim, drainage, etc.).

Finally, Oscar's utility over gardenspace and lawnspace is given by: $U_O(G,L_O) = 2G^{0.5} + L_O$. Maria's utility function is the same, except from the lawn-subspcript: $U_M(G,L_M) = 2G^{0.5} + L_M$. Notice that because gardenspace is a public good, G enters into both utility functions without a subscript – Oscar gets utility not just from the amount of garden he buys, G_O, but from the total amount bought by both of them, $G = G_O + G_M$, and the same goes for Maria.

a) Suppose Oscar, Maria, and both developers interact in competitive markets so that: Oscar and Maria buy lawn from the lawn developer at a given price p^L; Oscar and Maria buy garden from the garden developer at a given price p^G; and the lawn developer and garden developer buy land from Oscar and Maria at a given price w. We will now find competitive equilibrium prices and quantities. Normalize the equilibrium price of lawn, so $p^L = 1$. Write down the lawn developer's profit maximization problem as a function of its output L. To do this, you will need to 'substitute' the firm's input e^L out of the cost term using the production function. Derive the first order condition with respect to L. What is the equilibrium price of land w?

b) Using the equilibrium value of w, write down the garden developer's profit maximization problem as a function of G (you'll need to rearrange the production function a bit and 'substitute out' e^G from the cost term). The first order condition gives you the equilibrium price p^G as a function of the quantity of G produced and consumed in equilibrium. Since the garden developer is a price-taker, the first order condition just says that profits are maxed when the marginal revenue, p^G, is equal to the marginal cost of producing G. From part (a), what is the marginal revenue and marginal cost of producing L? The ratio of marginal costs, which equals p^G, is also this economy's MRT (where MRT is defined as the amount of lawn the economy needs to give up to produce one more unit of garden).

c) Write down Maria's budget constraint normalizing $p^L = 1$. What does Maria do with her endowment of vacant land? Using the budget equation, solve Maria's utility maximization problem. Do the same for Oscar. What is Maria's marginal utility from G? What is Maria's marginal utility from L? Give two interpretations for Maria's first order condition.

d) Since Oscar and Maria's consumer problems are identical, find the symmetric equilibrium values of G, G_O, G_M, and p^G. Using the budget equations, find L_O and L_M.

e) From Edgeworth box problems, when all goods are private goods, we know that competitive equilibrium allocations are Pareto efficient; starting from equilibrium, if person 1 sells a unit of good y for additional good x at the equilibrium price ratio, person 1 is left indifferent while person 2 is unaffected. Now consider the following. Suppose we are in competitive equilibrium, and Oscar sells a unit of L_O to the market for additional G_O at the equilibrium prices. Is Oscar better off, worse off or indifferent? What about Maria? Is the competitive equilibrium Pareto efficient? Explain.

f) Solve for the set of Pareto efficient allocations. To do this use the two utility functions, the production functions and the endowments to derive the Samuelson condition. The Samuelson condition is described in Professor Spiegel's 'public goods' handout; a version of it is also discussed on page 615 of the textbook. Is the efficient amount of gardenspace greater than, less than or equal to the competitive equilibrium amount?

6. Externalities

Two chemical factories, A and B, are located along a river. Firm A is upstream, Firm B is downstream. Therefore, the polluted water that A emits affects the productivity of Firm B. The firms use two inputs, water (W) and oil (Z), and produce a single output, according to the following production technologies:

Plant A: $q_A = W_A^{1/2} Z_A^{1/2}$
Plant B: $q_B = W_A^{-1} W_B^{1/2} Z_B^{1/2}$

The firms have to pay for the inputs they use. Input prices are p_W and p_Z, respectively. Total endowments of water is 3, total endowments of oil is 4/3.

a) Write down the profit functions of the two firms. Assume that the output price is normalized to 1.

b) Derive the equilibrium value of W_A (Hint: 1.Derive a relationship between p_W and p_Z, using the first-order conditions of Firm A. 2. Do the same, using now the first-order conditions of Firm B. 3. What is the only value of W_A which is compatible with both?)

c) Derive the equilibrium prices, p_W and p_Z (Hint: The demand functions for oil, Z_A and Z_B, should sum up to total endowment.)

d) Now you should be able to determine all equilibrium quantities.

e) Is this allocation Pareto-efficient? (Hint: The way to calculate Pareto-efficient allocations is the following. For any given output level of Firm B calculate the optimal input combination of Firm A. Can the equilibrium allocations derived in points b-d eventually coincide with a particular Pareto-efficient allocation? Remark: The algebraic manipulations you need are not difficult but quite long.)

f) What does this all have to do with the First Welfare Theorem?