Correction to Sect. 9

In Prop. 9d11 and Theorem 9d12, Jordan measurability of $f+g$ must be proved first.

In Prop. 9d11 this is easy: the bounded function $(f+g)_i = (f_i+g_i)_i$ is integrable on a Jordan set for all i; it follows that $f+g$ is Jordan measurable.

Theorem 9d12 needs more effort. We do it using the oscillation function.

Proposition. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are Jordan measurable then $f+g$ is Jordan measurable.

Exercise. f is Jordan measurable is and only if ${}^*\!\!\int_B \operatorname{Osc}_{f_i} = 0$ for all boxes $B \subset \mathbb{R}^n$ and all $i = 1, 2, \dots$; here $f_i = \operatorname{mid}(-i, f, i)$ is as in (9d10).

Prove it.

Exercise. $\operatorname{Osc}_{f_i} \uparrow \operatorname{Osc}_f$ (as $i \to \infty$).

Prove it.

Exercise. $\operatorname{Osc}_{f+g} \le \operatorname{Osc}_f + \operatorname{Osc}_g$.

Prove it.

Given $\varepsilon > 0$ and a box $B \subset \mathbb{R}^n$, we introduce sets

$$\begin{aligned} X &= \{x \in B : \operatorname{Osc}_{f+g}(x) \ge 2\varepsilon\} \,, \\ Y_i &= \{x \in B : \operatorname{Osc}_{f_i}(x) \ge \varepsilon\} \,, \\ Z_i &= \{x \in B : \operatorname{Osc}_{g_i}(x) \ge \varepsilon\} \,. \end{aligned}$$

Exercise. $X \subset \bigcup_i (Y_i \cup Z_i)$.

Prove it.

Exercise. Let $E_i, F_i \subset \mathbb{R}^n$ be Jordan sets, $E, F \subset \mathbb{R}^n$ sets, $E_i \downarrow E$, $F_i \uparrow F$, and $E \subset F$; then $\lim_i v(E_i) \le \lim_i v(F_i)$.

Prove it.[1]

Corollary. Let $F_i \subset \mathbb{R}^n$ be Jordan sets, $F \subset \mathbb{R}^n$ a set, $F_i \uparrow F$, and $K \subset \mathbb{R}^n$ a compact set, $K \subset F$; then $v^*(K) \le \lim_i v(F_i)$.

In particular, if $v(F_i) = 0$ for all i then $v^*(K) = 0$.

For example, every countable compact set is of volume zero.[2]

Proof of the Proposition. X is of volume zero, since it is a compact subset of the union of sets $Y_i \cup Z_i$ of volume zero. □

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[1] Hint: $E_i \setminus F_i \downarrow \emptyset$.

[2] By the way, can you imagine a countable compact set in general? (If you think you can, think again!)