11 From boundary to exterior derivative; Stokes' theorem

Terms "boundary" and "derivative" get new meaning, and become dual to each other.

11a What is the problem

A box $B \subset \mathbb{R}^n$ may be treated as a special case of a singular n-box in $\mathbb{R}^n$: $\Gamma : B \to \mathbb{R}^n$, $\Gamma(u) = u$. Thus every n-form ω on $\mathbb{R}^n$ leads to an additive box function

$$B \mapsto \int_B \omega = \int_B \omega(u, e_1, \dots, e_n)\, \mathrm{d}u$$

where $(e_1, \dots, e_n)$ is the usual orthonormal basis in $\mathbb{R}^n$. It is natural to define

(11a1) $$\int_E \omega = \int_E \omega(u, e_1, \dots, e_n)\, \mathrm{d}u$$

for all Jordan measurable sets $E \subset \mathbb{R}^n$. (In this sense, every n-form in $\mathbb{R}^n$ is locally proportional to the volume.)

The singular 2-box Γ of 10e2 is *not* a homeomorphism between the box $B = [0,1] \times [0, 2\pi] \subset \mathbb{R}^2$ and the disk $D = \{x : |x| \le 1\} \subset \mathbb{R}^2$. And nevertheless,

(11a2) $$\int_\Gamma \omega = \int_D \omega \quad \text{for every 2-form } \omega$$

since

$$\int_\Gamma \omega = \int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, (D_2\Gamma)_u\big)\,\mathrm{d}u = \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\,\omega\Big(\big(\begin{smallmatrix} r\cos\theta \\ r\sin\theta \end{smallmatrix}\big), \big(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\big), \big(\begin{smallmatrix} -r\sin\theta \\ r\cos\theta \end{smallmatrix}\big)\Big),$$

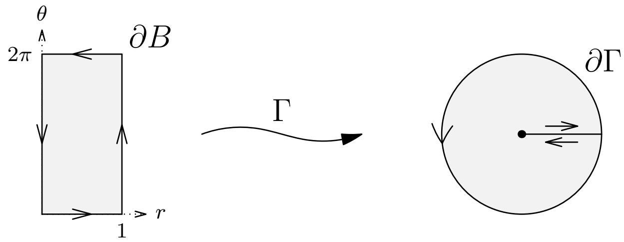
$$\int_D \omega = \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\,\omega\Big(\big(\begin{smallmatrix} r\cos\theta \\ r\sin\theta \end{smallmatrix}\big), \big(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\big)\Big),$$

and

$$L\Big(\big(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\big), \big(\begin{smallmatrix} -r\sin\theta \\ r\cos\theta \end{smallmatrix}\big)\Big) = rL\Big(\big(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\big)\Big)$$

for every antisymmetric bilinear form L on $\mathbb{R}^2$ (think, why). The missing segment $\{0\} \times [0,1]$ does not matter for the 2-dimensional integral.

We may say that this singular box is equivalent to the disk (w.r.t. 2-forms). However, what happens to the boundary? The boundary ∂B of B is not a box but the union of four 1-dimensional boxes, and $\Gamma|_{\partial B}$ may be treated as a path $\partial\Gamma$ consisting of four singular 1-boxes (one degenerated to a point).

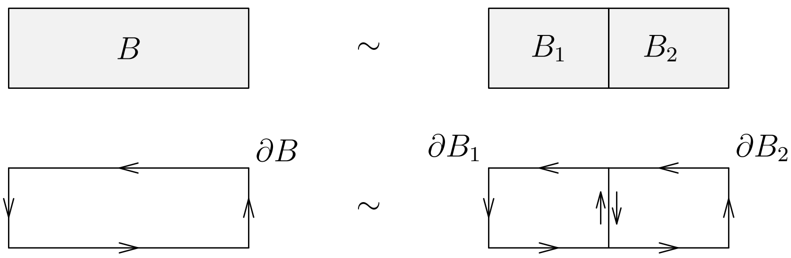


Interestingly,

(11a3) $$\int_{\partial\Gamma} \omega = \int_S \omega \quad \text{for every 1-form } \omega\,;$$

here $\int_S \omega = \int_0^{2\pi} \omega\big(\big(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\big), \big(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\big)\big)\,\mathrm{d}\theta$. The segment $\{0\}\times[0,1]$ does not harm since it occurs twice, with opposite signs. We may say that the boundary of this singular box is equivalent to the boundary of the disk.

Just a good luck? No! Rather, a manifestation of a deep and important relation between singular boxes and their boundaries. Another example:



11b Chains

11b1 Definition. A (singular) k-chain (in $\mathbb{R}^n$) is a formal linear combination of singular k-boxes.

That is,

$$C = c_1\Gamma_1 + \dots + c_p\Gamma_p \,,$$

where $a_1, \dots, a_p \in \mathbb{R}$ and $\Gamma_1, \dots, \Gamma_p$ are singular k-boxes. More formally, this is a real-valued function with finite support on the (huge!) set of all singular k-boxes;

$$c_1 = C(\Gamma_1), \dots, c_p = C(\Gamma_p) \,; \quad C(\Gamma) = 0 \text{ for all other } \Gamma \,.$$

Clearly, all k-chains are a (huge) vector space, with a basis indexed by all singular k-boxes. Less formally we say that the singular k-boxes *are* the basis, and each singular box is (a special case of) a chain: $\Gamma = 1 \cdot \Gamma$.

11b2 Definition.

$$\int_C \omega = c_1 \int_{\Gamma_1} \omega + \dots + c_p \int_{\Gamma_p} \omega$$

for every k-chain $C = c_1\Gamma_1 + \dots + c_p\Gamma_p$ and every k-form ω.

Note that the integral is bilinear; $\int_C \omega$ is linear in C for every ω (by construction), and linear in ω for every C (since $\int_\Gamma \omega$ evidently is linear in ω).

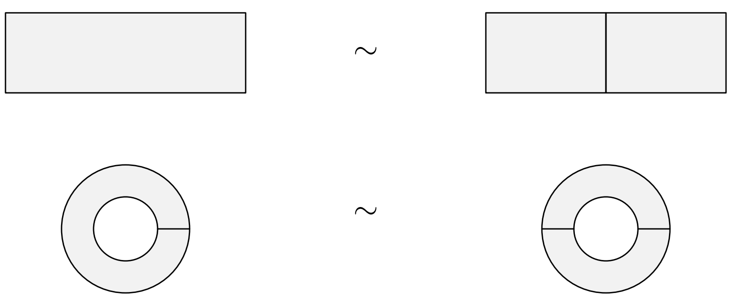
11b3 Definition. Two k-chains C_1, C_2 are *equivalent* if

$$\int_{C_1} \omega = \int_{C_2} \omega \quad \text{for all } k\text{-forms } \omega \text{ (of class } C^0) \,.$$

Let $B \subset \mathbb{R}^k$ be a box, P its partition, and $\Gamma : B \to \mathbb{R}^n$ a singular box. Then

$$\Gamma \sim \sum_{b \in P} \Gamma|_b \,,$$

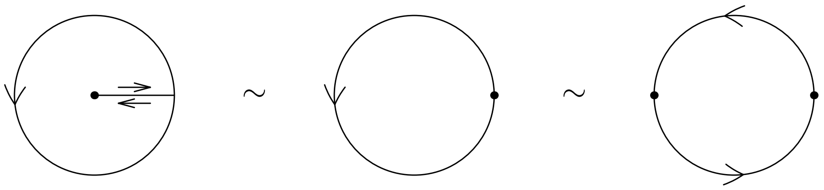
since $\Gamma \mapsto \int_\Gamma \omega$ is an additive function of a singular box.

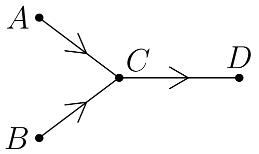


Recall that singular 1-boxes are C^1-paths.

By 10c12, equivalent paths are equivalent 1-chains.

By 10c10, the 1-chain $\gamma + \gamma_{-1}$ is equivalent to 0; here γ_{-1} is the inverse path.





$$ACD + BC \sim AC + BCD \sim AC + BC + CD$$

11c Order 0 and order 1

The case $k = 0$ is included as follows. The space $\mathbb{R}^0$ consists, by definition, of a single point 0. The only 0-dimensional box is $\{0\}$. A singular 0-box in $\mathbb{R}^n$ is thus $\{x\}$ for some $x \in \mathbb{R}^n$.[1] A 0-form on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \to \mathbb{R}$ (of class C^m). And

$$\int_{\{x\}} \omega = \omega(x) \,,$$

of course. Accordingly, $\int_C \omega = c_1\omega(x_1) + \dots + c_p\omega(x_p)$ for a 0-chain $C = c_1\{x_1\} + \dots + c_p\{x_p\}$.

11c1 Exercise. If two 0-chains are equivalent then they are equal.
Prove it.

The *boundary* of a singular 1-box $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is, by definition, the 0-chain

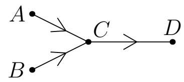
$$\partial\gamma = \{\gamma(t_1)\} - \{\gamma(t_0)\} \,,$$

a linear combination of two singular 0-boxes (not to be confused with $\gamma(t_1) - \gamma(t_0)$). Thus,

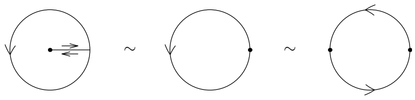
$$\int_{\partial\gamma} \omega = \omega(\gamma(t_1)) - \omega(\gamma(t_0)) \quad \text{for a 0-form } \omega \,.$$

[1] Well, more formally, it is $\{(0, x)\}$.

The boundary of a 1-chain $C = c_1\gamma_1 + \dots + c_p\gamma_p$ is, by definition, the 0-chain $\partial C = c_1\partial\gamma_1 + \dots + c_p\partial\gamma_p$. For example,

the boundary of 

is $-\{A\} - \{B\} + \{C\} + \{D\}$;

the boundary of 
is 0.

Note that the map $C \mapsto \partial C$ is linear (by construction).

Given a 0-form ω of class C^1 on $\mathbb{R}^n$, that is, a continuously differentiable function $\omega : \mathbb{R}^n \to \mathbb{R}$, its derivative $D\omega$ may be thought of as a 1-form of class C^0 on $\mathbb{R}^n$, denoted $d\omega$;

(11c2) $$(d\omega)(x,h) = (D\omega)_x(h) = (D_h\omega)_x \, .$$

11c3 Proposition. (*Stokes' theorem for $k = 1$*)
Let C be a 1-chain in $\mathbb{R}^n$, and ω a 0-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega \, .$$

Proof. By linearity in C it is sufficient to prove it for $C = \gamma$ (a single 1-box, that is, a path $\gamma : [t_0, t_1] \to \mathbb{R}^n$). We have

$$\int_\gamma d\omega = \int_{t_0}^{t_1} d\omega\big(\gamma(t), \gamma'(t)\big) \, \mathrm{d}t = \int_{t_0}^{t_1} (D\omega)_{\gamma(t)}(\gamma'(t)) \, \mathrm{d}t =$$
$$= \int_{t_0}^{t_1} \Big(\frac{\mathrm{d}}{\mathrm{d}t} \omega(\gamma(t)) \Big) \, \mathrm{d}t = \omega(\gamma(t_1)) - \omega(\gamma(t_0)) = \int_{\partial\gamma} \omega \, .$$

□

11c4 Corollary.
$$C_1 \sim C_2 \quad \text{implies} \quad \partial C_1 = \partial C_2$$
for arbitrary 1-chains C_1, C_2 in $\mathbb{R}^n$.

Indeed, $\int_{\partial C_1} \omega = \int_{C_1} d\omega = \int_{C_2} d\omega = \int_{\partial C_2} \omega$ for every 0-form ω of class C^1. Similarly to 11c1 it follows that $\partial C_1 = \partial C_2$.

The case $k = 1$ is special; for higher k we'll see that $C_1 \sim C_2$ implies $\partial C_1 \sim \partial C_2$ but not $\partial C_1 = \partial C_2$. Nothing like 11c1 exists for higher k.

Let us try to prove that $C_1 \sim C_2 \implies \partial C_1 \sim \partial C_2$ for $k = 1$ without 11c1. The only problem is that $C^1(\mathbb{R}^n) \ne C^0(\mathbb{R}^n)$. However, $C^1(\mathbb{R}^n)$ is dense in $C^0(\mathbb{R}^n)$ in the following sense.

11c5 Lemma. For every $f \in C^0(\mathbb{R}^n)$ there exist $f_i \in C^1(\mathbb{R}^n)$ such that $f_i \to f$ uniformly on bounded sets.

Proof (sketch, for $n = 2$). Define f_ε for $\varepsilon > 0$ by

$$f_\varepsilon(x_1, x_2) = \frac{1}{\varepsilon^2} \int_{[x_1, x_1+\varepsilon]\times[x_2, x_2+\varepsilon]} f \, ,$$

then the partial derivative

$$\frac{\partial}{\partial x_1} f_\varepsilon(x_1, x_2) = \frac{1}{\varepsilon^2} \int_{[x_2, x_2+\varepsilon]} \big(f(x_1+\varepsilon, \cdot) - f(x_1, \cdot) \big)$$

is continuous; similarly, the other partial derivative is continuous; thus, $f_\varepsilon \in C^1(\mathbb{R}^n)$. The uniform convergence to f (as $\varepsilon \to 0$) follows from uniform continuity of f (on bounded sets). □

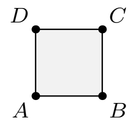
11c6 Exercise. Complete the proof, and generalize it to all dimensions.

Thus, C^0 may be replaced with C^1 in Def. 11b3 for $k = 0$.

11d Order 1 and order 2: exterior derivative

The boundary of a singular 2-box Γ is, by definition, the 1-chain

$$\Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD} \, .$$



This is not really a definition of a 1-chain, since I did not specify the four 1-dimensional boxes (which is very easy to do); but its equivalence class is well-defined, and this is all we need solving the following question.

Given a 1-form ω, can we construct a 2-form, call it $d\omega$, such that $\int_C d\omega = \int_{\partial C} \omega$ for all 2-chains C?

We have a function $C \mapsto \int_{\partial C} \omega$ of a singular box; this is an additive function, since the map $\Gamma \mapsto \partial\Gamma$ is additive (up to equivalence).

We want to differentiate this additive function in the hope that its derivative exists and is a 2-form $d\omega$.

Note that

(11d1) $$\partial(\partial\Gamma) = 0 \quad \text{for a singular 2-box } \Gamma\,;$$

by 11c3, $\int_{\partial\Gamma} d\omega = \int_{\partial(\partial\Gamma)} \omega = 0$ for every 0-form ω. It should be $\int_\Gamma d(d\omega) = \int_{\partial\Gamma} d\omega = 0$ for all Γ, that is, $d(d\omega) = 0$. Indeed, this fact will be proved, see (11e4). A wonder: the second derivative of a 0-form is always zero, irrespective of the second derivatives of the function! Indeed, exterior derivative is very similar to the usual derivative for 0-forms, but very dissimilar for 1-forms.

For now we only need to *guess* a formula for $d\omega$; having the formula, hopefully we'll be able to prove the equality.

Given a point $x \in \mathbb{R}^n$ and two vectors $h, k \in \mathbb{R}^n$, we consider small singular boxes $\Gamma_\varepsilon : [0,1] \times [0,1] \to \mathbb{R}^n$,

$$\Gamma_\varepsilon(u_1, u_2) = x + \varepsilon u_1 h + \varepsilon u_2 k\,;$$

an additive function on Γ_ε should be of order ε^2 as $\varepsilon \to 0+$; we divide it by ε^2 and calculate the limit:

$$\frac{1}{\varepsilon^2}\int_{\partial\Gamma_\varepsilon} \omega = \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_1 h, \varepsilon h)\,\mathrm{d}u_1 + \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon h+\varepsilon u_2 k, \varepsilon k)\,\mathrm{d}u_2 -$$
$$- \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_1 h+\varepsilon k, \varepsilon h)\,\mathrm{d}u_1 - \frac{1}{\varepsilon^2}\int_0^1 \omega(x+\varepsilon u_2 k, \varepsilon k)\,\mathrm{d}u_2 =$$
$$= \int_0^1 \frac{\omega(x+\varepsilon u_1 h, h) - \omega(x+\varepsilon u_1 h+\varepsilon k, h)}{\varepsilon}\,\mathrm{d}u_1 +$$
$$+ \int_0^1 \frac{\omega(x+\varepsilon h+\varepsilon u_2 k, k) - \omega(x+\varepsilon u_2 k, k)}{\varepsilon}\,\mathrm{d}u_2 \to -\big(D_k\omega(\cdot,h)\big)_x + \big(D_h\omega(\cdot,k)\big)_x\,.$$

Taking into account that

$$\frac{1}{\varepsilon^2}\int_{\Gamma_\varepsilon} d\omega \to (d\omega)(x,h,k)$$

(for arbitrary 2-form $d\omega$) we see that the needed $d\omega$ (if exists) is as follows.

11d2 Definition. The *exterior derivative* of a 1-form ω of class C^1 is a 2-form $d\omega$ defined by

$$(d\omega)(\cdot,h,k) = D_h\omega(\cdot,k) - D_k\omega(\cdot,h)\,.$$

11d3 Theorem. (*Stokes' theorem for $k = 2$*)

Let C be a 2-chain in $\mathbb{R}^n$, and ω a 1-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega\,.$$

The proof will be given in Sect. 11g.

11e Algebra of differential forms

For every $k = 0, 1, \dots, n$ all k-forms (of class C^m) on $\mathbb{R}^n$ are a vector space. For $k = 0$ this space is just $C^m(\mathbb{R}^n)$.

The product $f\omega$ of a 0-form f and a k-form ω is another k-form $f\omega$ defined by

$$(f\omega)(x, h_1, \dots, h_k) = f(x)\omega(x, h_1, \dots, h_k) \quad \text{for } x, h_1, \dots, h_k \in \mathbb{R}^n\,;$$

it is of class C^m whenever f and ω are; the mapping $(f, \omega) \mapsto f\omega$ is bilinear; also, $g(f\omega) = (gf)\omega$.

The *exterior derivative* of a 0-form $f \in C^1(\mathbb{R}^n)$ is a 1-form df defined by (recall (11c2))

$$(df)(x, h) = (Df)_x(h) = (D_h f)_x\,; \tag{11e1}$$

the mapping $f \mapsto df$ is linear; also, $d(fg) = f\,dg + g\,df$.

The exterior derivative of the i-th coordinate function $x \mapsto x_i$ is traditionally denoted by dx_i (for $i = 1, \dots, n$); thus,

$$(dx_i)(x, h) = h_i \qquad \text{for all } x \in \mathbb{R}^n \text{ and } h = (h_1, \dots, h_n) \in \mathbb{R}^n\,. \tag{11e2}$$

A linear form L on $\mathbb{R}^n$ is generally $L(h) = \sum_{i=1}^n c_i h_i$ for some $c_1, \dots, c_n \in \mathbb{R}$; thus, a 1-form ω on $\mathbb{R}^n$ is generally $\omega(x, h) = \sum_{i=1}^n f_i(x) h_i$ for some $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}$. That is (recall Sect. 10c),

$$\omega = \sum_{i=1}^n f_i\,dx_i\,;$$

ω is of class C^m if and only if all f_i are. In particular,

$$df = \sum_{i=1}^n D_i f\,dx_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dx_i\,,$$

since $D_h f = \sum_{i=1}^n (D_i f) h_i$.

A 1-form on $\mathbb{R}^1$ is $f\,dx_1$. Treating a box $B \subset \mathbb{R}^1$ as a singular 1-box $(\mathrm{id} : B \to \mathbb{R}^1)$ we have $\int_B \omega = \int_B f(x_1)\,\mathrm{d}x_1$ for $\omega = f\,dx_1$, since $(dx_1)(x, e_1) = 1$ (recall (11a1) and (11e2)).

The *exterior* (or wedge) *product* of two 1-forms ω_1, ω_2 is a 2-form $\omega_1 \wedge \omega_2$ defined by[1]

$$(\omega_1 \wedge \omega_2)(x, h, k) = \omega_1(x, h)\omega_2(x, k) - \omega_1(x, k)\omega_2(x, h)\,;$$

[1] Why $dx_i \wedge dx_j$ rather than $dx_i\,dx_j$? In fact, both notations are in use; the wedge symbol "$\wedge$" helps us remember that this operation is antisymmetric.

it is of class C^m whenever ω_1 and ω_2 are; the mapping $(\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$ is bilinear and antisymmetric: $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$. Also, $(f\omega_1) \wedge (g\omega_2) = (fg)(\omega_1 \wedge \omega_2)$. By (11e2),

(11e3) $$(dx_i \wedge dx_j)(x, h, k) = h_i k_j - k_i h_j = \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix} .$$

A bilinear form L on $\mathbb{R}^n$ is generally

$$L(h,k) = \sum_{i=1}^n \sum_{j=1}^n c_{i,j} h_i k_j ;$$

it is antisymmetric if and only if $c_{i,j} = -c_{j,i}$; in this case $L(h,k) = \sum_{i,j} c_{i,j} h_i k_j = \sum_{i<j} c_{i,j}(h_i k_j - h_j k_i)$. Thus, by (11e3), a 2-form ω on $\mathbb{R}^n$ is generally

$$\omega = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j = \frac{1}{2} \sum_{i,j} f_{i,j}\, dx_i \wedge dx_j ;$$

in the former notation $f_{i,j}$ are given for $i < j$ only, while in the latter notation $f_{i,j} = -f_{j.i}$; ω is of class C^m if and only if all $f_{i,j}$ are. For example, the 2-form of 10e12 is $x_1\, dx_2 \wedge dx_3$.

A 2-form on $\mathbb{R}^2$ is $f\, dx_1 \wedge dx_2$. Treating a box $B \subset \mathbb{R}^2$ as a singular 2-box $(\mathrm{id} : B \to \mathbb{R}^2)$ we have $\int_B f\, dx_1 \wedge dx_2 = \int_B f(x)\, \mathrm{d}x_1 \mathrm{d}x_2$, since $(dx_1 \wedge dx_2)(x, e_1, e_2) = 1$ (recall (11a1) and (11e3)).

We turn to Def. 11d2. Let $\omega = df$, $f \in C^2(\mathbb{R}^n)$; then, by (11e1) (and Sect. 2g), $(d\omega)(\cdot, h, k) = D_h \omega(\cdot, k) - D_k \omega(\cdot, h) = D_h(D_k f) - D_k(D_h f) = 0$, that is,

(11e4) $$d(df) = 0 ,$$

as it should be (recall (11d1) and the paragraph after it).

Now consider $d(f\omega)$ for $f \in C^1(\mathbb{R}^n)$ and a 1-form ω of class C^1 on $\mathbb{R}^n$. We have

$$\begin{aligned}\big(d(f\omega)\big)(\cdot, h, k) &= D_h\big(f\omega(\cdot, k)\big) - D_k\big(f\omega(\cdot, h)\big) = \\ &= (D_h f)\omega(\cdot, k) + f D_h \omega(\cdot, k) - (D_k f)\omega(\cdot, h) - f D_k \omega(\cdot, h) = \\ &= f d\omega(\cdot, h, k) + (D_h f)\omega(\cdot, k) - (D_k f)\omega(\cdot, h) = \\ &= f d\omega(\cdot, h, k) + df(\cdot, h)\omega(\cdot, k) - df(\cdot, k)\omega(\cdot, h) = \\ &= f d\omega(\cdot, h, k) + (df \wedge \omega)(\cdot, h, k) ;\end{aligned}$$

thus,

(11e5) $$d(f\omega) = df \wedge \omega + f\, d\omega .$$

It follows via (11e4) that

$$d(f\,dg) = df \wedge dg \tag{11e6}$$

for $f \in C^1(\mathbb{R}^n)$, $g \in C^2(\mathbb{R}^n)$, and we get the following definition equivalent to 11d2.

11e7 Definition. The *exterior derivative* of a 1-form ω of class C^1 is a 2-form $d\omega$ defined by

$$d\omega = \sum_{i=1}^n df_i \wedge dx_i \quad \text{for } \omega = \sum_{i=1}^n f_i\,dx_i\,.$$

The 2-form $d\omega$ is of class C^m whenever ω is of class C^{m+1}; the mapping $\omega \mapsto d\omega$ is linear; and $d(f\omega)$ is given by (11e5).

11e8 Exercise. Check that

$$\int_\Gamma \omega = \int_B \sum_{i<j} f_{i,j}(x) \frac{\partial(x_i,x_j)}{\partial(u_1,u_2)}\,\mathrm{d}u_1\mathrm{d}u_2$$

for every 2-form $\omega = \sum_{i<j} f_{i,j}\,dx_i \wedge dx_j$ on $\mathbb{R}^n$ and singular 2-box $\Gamma : B \to \mathbb{R}^n$; here $x = (x_1,\dots,x_n) = \Gamma(u_1,u_2)$ and

$$\frac{\partial(x_i,x_j)}{\partial(u_1,u_2)} = \begin{vmatrix} \frac{\partial x_i}{\partial u_1} & \frac{\partial x_i}{\partial u_2} \\ \frac{\partial x_j}{\partial u_1} & \frac{\partial x_j}{\partial u_2} \end{vmatrix}\,.$$

In particular,

$$\int_\Gamma dx_i \wedge dx_j = \int_B \frac{\partial(x_i,x_j)}{\partial(u_1,u_2)}\,\mathrm{d}u_1\mathrm{d}u_2\,.$$

11e9 Exercise. [1] (a) Let $\Gamma : B \to \mathbb{R}^3$ be a singular 2-box in $\mathbb{R}^3$, and $\Gamma_0 : B \to \mathbb{R}^3$ its projection onto the xy plane; that is, $\Gamma(u) = \big(\Gamma_1(u),\Gamma_2(u),\Gamma_3(u)\big)$ and $\Gamma_0(u) = \big(\Gamma_1(u),\Gamma_2(u),0\big)$ for $u \in B$. Prove that $\int_\Gamma dx \wedge dy = \int_{\Gamma_0} dx \wedge dy$.

(b) Consider $\Gamma : [0,a] \times [0,\pi] \to \mathbb{R}^3$, $\Gamma(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$. Sketch the surface noting that θ varies from 0 to π, not from 0 to 2π. Try to determine $\int_\Gamma dx \wedge dy$ by geometrical reasoning, and then check your answer by integration. Do the same for $dy \wedge dz$ and $dz \wedge dx$.

11e10 Exercise. [2] (a) Integrate a 2-form $x\,dy \wedge dz + y\,dx \wedge dy$ on $\mathbb{R}^3$ over the singular 2-box $\Gamma : [0,1] \times [0,1] \to \mathbb{R}^3$, $\Gamma(u,v) = (u+v, u^2 - v^2, uv)$.

(b) The same for $\Gamma : [0,2\pi] \times [0,1] \to \mathbb{R}^3$, $\Gamma(u,v) = (v\cos u, v\sin u, u)$.

[1] Shurman, Ex. 9.5.1

[2] Shurman, Ex. 9.5.2

11e11 Exercise. [1] (a) Calculate $(a_1\,dx_1 + a_2\,dx_2) \wedge (b_1\,dx_1 + b_2\,dx_2)$, observe a 2×2 determinant;

(b) calculate $(a_1\,dx_1 + a_2\,dx_2 + a_3\,dx_3) \wedge (b_1\,dx_1 + b_2\,dx_2 + b_3\,dx_3)$, observe a cross product.

11e12 Exercise. Check that $d(x\,dy - y\,dx) = 2\,dx \wedge dy$.

11f Change of variables

Given a mapping $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$, every singular k-box $\Gamma : B \to \mathbb{R}^\ell$ leads to a singular k-box $\varphi \circ \Gamma : B \to \mathbb{R}^n$. Thus, every k-form ω on $\mathbb{R}^n$ leads to a box function $\Gamma \mapsto \int_{\varphi\circ\Gamma} \omega$; it is additive (since the mapping $\Gamma \mapsto \varphi \circ \Gamma$ is). Can we find a k-form $\varphi^*\omega$ on $\mathbb{R}^\ell$ such that $\int_{\varphi\circ\Gamma} \omega = \int_\Gamma \varphi^*\omega$ for all Γ?

11f1 Definition. Given a k-form ω on $\mathbb{R}^n$ and a mapping $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$, the *pullback* of ω along φ is a k-form $\varphi^*\omega$ on $\mathbb{R}^\ell$ defined by

$$\begin{aligned}(\varphi^*\omega)(x, h_1, \dots, h_k) &= \omega\big(\varphi(x), (D\varphi)_x(h_1), \dots, (D\varphi)_x(h_k)\big) = \\ &= \omega\big(\varphi(x), (D_{h_1}\varphi)_x, \dots, (D_{h_k}\varphi)_x\big) \quad \text{for } x, h_1, \dots, h_k \in \mathbb{R}^\ell \,.\end{aligned}$$

The form $\varphi^*\omega$ is of class C^m whenever ω is of class C^m and φ is of class C^{m+1}. The mapping $\omega \mapsto \varphi^*\omega$ is linear. For $k = 0$ the pullback is just the composition: $(\varphi^* f)(x) = f(\varphi(x))$; $\varphi^* f = f \circ \varphi$ (no need in C^{m+1} in this case). And $\varphi^*(f\omega) = (\varphi^* f)(\varphi^*\omega) = (f \circ \varphi)\varphi^*\omega$ for $f \in C^1(\mathbb{R}^n)$.

A singular k-box Γ in $\mathbb{R}^n$ is a C^1-mapping $B \to \mathbb{R}^n$ on a box $B \subset \mathbb{R}^k$ rather than the whole $\mathbb{R}^k$, but still, the pullback $\Gamma^*\omega$ is well-defined (on B),

$$(\Gamma^*\omega)(u, h_1, \dots, h_k) = \omega\big(\Gamma(u), (D_{h_1}\Gamma)_u, \dots, (D_{h_k}\Gamma)_u\big)$$

for $u \in B$ and $h_1, \dots, h_k \in \mathbb{R}^k$. In particular, for the usual basis $e_1, \dots, e_k$ of $\mathbb{R}^k$ we have $(\Gamma^*\omega)(u, e_1, \dots, e_k) = \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)$. Thus, the definition of $\int_\Gamma \omega$ given in Sect. 10e may be rewritten as $\int_\Gamma \omega = \int_B (\Gamma^*\omega)(u, e_1, \dots, e_k)\,du$. Using (11a1) we get

$$\int_\Gamma \omega = \int_B \Gamma^*\omega \,. \tag{11f2}$$

We see that it was the integral of the pullback, from the very beginning!

By the chain rule 2b12,

$$\big(D(\varphi \circ \Gamma)\big)_u = (D\varphi)_{\Gamma(u)} \circ (D\Gamma)_u \,;$$

[1] Shurman, Sect. 9.7]

thus,

$$\begin{aligned}\big((\varphi\circ\Gamma)^*\omega\big)(u,h_1,\dots,h_k) &= \omega\big((\varphi\circ\Gamma)(u),(D(\varphi\circ\Gamma))_u(h_1),\dots,(D(\varphi\circ\Gamma))_u(h_k)\big) = \\ &= \omega\big(\varphi(\Gamma(u)),(D\varphi)_{\Gamma(u)}(D\Gamma)_u h_1,\dots,(D\varphi)_{\Gamma(u)}(D\Gamma)_u h_k\big) = \\ &= (\varphi^*\omega)\big(\Gamma(u),(D\Gamma)_u h_1,\dots,(D\Gamma)_u h_k\big) = \big(\Gamma^*(\varphi^*\omega)\big)(u,h_1,\dots,h_k)\,,\end{aligned}$$

that is,[1]

$$(\varphi\circ\Gamma)^*\omega = \Gamma^*(\varphi^*\omega)\,,$$

which leads to the change of variable formula

$$\int_{\varphi\circ\Gamma}\omega = \int_B(\varphi\circ\Gamma)^*\omega = \int_B\Gamma^*(\varphi^*\omega) = \int_\Gamma\varphi^*\omega$$

for singular boxes, and therefore (by linearity in C), also for k-chains C is $\mathbb{R}^n$:

$$\int_{\varphi\circ C}\omega = \int_C\varphi^*\omega\,, \tag{11f3}$$

where $\varphi\circ C = c_1(\varphi\circ\Gamma_1)+\dots+c_p(\varphi\circ\Gamma_p)$ for $c = c_1\Gamma_1+\dots+c_p\Gamma_p$.

11f4 Lemma. For every 0-form $f\in C^1(\mathbb{R}^n)$ and $\varphi\in C^1(\mathbb{R}^\ell\to\mathbb{R}^n)$,

$$\varphi^*(df) = d(\varphi^*f)\,.$$

Proof.

$$\begin{aligned}\big(\varphi^*(df)\big)(x,h) &= (df)\big(\varphi(x),(D\varphi)_x h\big) = \\ &= (Df)_{\varphi(x)}(D\varphi)_x h \overset{2b12}{=} D(f\circ\varphi)_x h = d(\varphi^*f)(x,h)\,.\end{aligned}$$

□

11f5 Lemma. For all 1-forms ω_1,ω_2 on $\mathbb{R}^n$ and $\varphi\in C^1(\mathbb{R}^\ell\to\mathbb{R}^n)$,

$$\varphi^*(\omega_1\wedge\omega_2) = (\varphi^*\omega_1)\wedge(\varphi^*\omega_2)\,.$$

Proof.

$$\begin{aligned}&\big(\varphi^*(\omega_1\wedge\omega_2)\big)(x,h,k) = (\omega_1\wedge\omega_2)\big(\varphi(x),(D\varphi)_x h,(D\varphi)_x k\big) = \\ &= \omega_1\big(\varphi(x),(D\varphi)_x h\big)\omega_2\big(\varphi(x),(D\varphi)_x k\big) - \omega_1\big(\varphi(x),(D\varphi)_x k\big)\omega_2\big(\varphi(x),(D\varphi)_x h\big) = \\ &= (\varphi^*\omega_1)(x,h)(\varphi^*\omega_2)(x,k) - (\varphi^*\omega_1)(x,k)(\varphi^*\omega_2)(x,h) = \\ &= \big((\varphi^*\omega_1)\wedge(\varphi^*\omega_2)\big)(x,h,k)\,.\end{aligned}$$

□

[1] The same argument gives a more general formula $(\varphi\circ\psi)^*\omega = \psi^*(\varphi^*\omega)$.

11f6 Lemma. For every 1-form ω of class C^1 on $\mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^\ell \to \mathbb{R}^n)$,

$$\varphi^*(d\omega) = d(\varphi^*\omega)\,.$$

Proof. We have $\omega = \sum_{i=1}^n f_i\,dx_i$ and $d\omega = \sum_{i=1}^n df_i \wedge dx_i$. It is sufficient to prove that $\varphi^*(df_i \wedge dx_i) = d\big(\varphi^*(f_i\,dx_i)\big)$. We have

$$\varphi^*(df_i \wedge dx_i) \overset{11f5}{=} \varphi^*(df_i) \wedge \varphi^*(dx_i) \overset{11f4}{=}$$
$$= d(\varphi^* f_i) \wedge d(\varphi^* x_i) \overset{11e6}{=} d\big(\varphi^*(f_i)\,d\varphi^*(x_i)\big) \overset{11f4}{=} d\big(\varphi^*(f_i)\varphi^*(dx_i)\big) = d\big(\varphi^*(f_i\,dx_i)\big)\,.$$

□

A differential form may be defined on an open subset of $\mathbb{R}^n$ (rather than the whole $\mathbb{R}^n$); everything generalizes readily to this case. Below, in some exercises, some forms are defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

11f7 Exercise. [1] (a) $(x,y) = \varphi(r,\theta) = (r\cos\theta, r\sin\theta)$; find $\varphi^*\omega$ for $\omega = dx \wedge dy$;

(b) the same φ, but $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$;

(c) the same ω as in (b), but $(x,y) = \varphi(u,v) = (u^2 - v^2, 2uv)$.

11f8 Exercise. [2] Consider mappings: $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$, $\psi(u,v) = (u^2 - v^2, 2uv)$, and $\xi(r,\theta) = (r^2, 2\theta)$. For $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ find $\varphi^*\omega$, $\xi^*(\varphi^*\omega)$, $\psi^*\omega$, and $\varphi^*(\psi^*\omega)$. Explain the result.

11f9 Exercise. [3] For a given $r > 0$ consider a singular 2-box $\Gamma : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$, $\Gamma(\theta,\varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$ and a 2-form $\omega = -\frac{x}{r}\,dy \wedge dz - \frac{y}{r}\,dz \wedge dx - \frac{z}{r}\,dx \wedge dy$. Find the pullback $\Gamma^*\omega$.

11g Proving the theorem

11g1 Exercise. Let $\Gamma, \Gamma_1, \Gamma_2, \dots : B \to \mathbb{R}^n$ be singular k-boxes such that $\Gamma_i \to \Gamma$ in C^1, that is,

$$\Gamma_i \to \Gamma\,, \quad D_1\Gamma_i \to D_1\Gamma\,, \quad \dots\,, \quad D_k\Gamma_i \to D_k\Gamma \quad \text{uniformly on } B\,.$$

Then

$$\int_{\Gamma_i} \omega \to \int_\Gamma \omega \quad \text{for every } k\text{-form } \omega \text{ on } \mathbb{R}^n.$$

Prove it.

---

[1] Shurman, Sect. 9.9.

[2] Shurman, Sect. 9.9.

[3] Shurman, Ex. 9.9.4.

11g2 Exercise. Let $\Gamma, \Gamma_1, \Gamma_2, \dots : B \to \mathbb{R}^n$ be singular 2-boxes such that $\Gamma_i \to \Gamma$ in C^1. Then

$$\int_{\partial\Gamma_i} \omega \to \int_{\partial\Gamma} \omega \quad \text{for every 1-form } \omega \text{ on } \mathbb{R}^n.$$

Prove it.

11g3 Lemma. For every $\Gamma \in C^1(B \to \mathbb{R}^n)$ there exist $\Gamma_i \in C^2(B \to \mathbb{R}^n)$ such that $\Gamma_i \to \Gamma$ in C^1.

Proof (sketch, for $B = [0,1] \times [0,1] \subset \mathbb{R}^2$). The argument of 11c5 needs only a slight modification. We define Γ_ε for $\varepsilon > 0$ by

$$\Gamma_\varepsilon(u_1, u_2) = \frac{1}{\varepsilon^2} \int_{[u_1, u_1+\varepsilon] \times [u_2, u_2+\varepsilon]} \Gamma\Big(\frac{v_1}{1+\varepsilon}, \frac{v_2}{1+\varepsilon} \Big) ,$$

then the partial derivative

$$\frac{\partial}{\partial u_1} \Gamma_\varepsilon(u_1, u_2) = \frac{1}{\varepsilon} \int_{[u_2, u_2+\varepsilon]} \frac{1}{\varepsilon} \Big(\Gamma\Big(\frac{u_1+\varepsilon}{1+\varepsilon}, \frac{v_2}{1+\varepsilon} \Big) - \Gamma\Big(\frac{u_1}{1+\varepsilon}, \frac{v_2}{1+\varepsilon} \Big) \Big) \, \mathrm{d}v_2$$

is of class C^1 and converges (uniformly) to $\frac{\partial}{\partial u_1} \Gamma(u_1, u_2)$. □

Proof of Theorem 11d3. It is sufficient to prove the equality $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ for every singular 2-box Γ. Applying (11f2) to the 2-box B and the four 1-boxes constituting ∂B we transform the needed equality into $\int_B \Gamma^*(d\omega) = \int_{\partial B} \Gamma^* \omega$. By 11g1, 11g2 and 11g3 we may assume that Γ is of class C^2. Thus, 11f6 applies, and the needed equality becomes

$$\int_B d(\Gamma^* \omega) = \int_{\partial B} \Gamma^* \omega \, .$$

Now we may forget the singular 2-box Γ in $\mathbb{R}^n$ and the 1-form ω on $\mathbb{R}^n$; it remains to prove the equality $\int_B d\omega = \int_{\partial B} \omega$ for every 1-form ω of class C^1 on the square $B = [0,1] \times [0,1] \subset \mathbb{R}^2$.

In general $\omega = f_1 \, du_1 + f_2 \, du_2$; by linearity in ω we may consider two 1-forms separately, $f_1 \, du_1$ and $f_2 \, du_2$; we consider only $\omega = f(u_1, u_2) \, du_1$, since the other case is similar.

We have $d\omega = df \wedge du_1 = \big(\frac{\partial f}{\partial u_1} \, du_1 + \frac{\partial f}{\partial u_2} \, du_2 \big) \wedge du_1 = -\frac{\partial f}{\partial u_2} \, du_1 \wedge du_2$, thus

$$\int_B d\omega = -\int_{[0,1]\times[0,1]} \frac{\partial f}{\partial u_2} \, du_1 du_2 = -\int_0^1 du_1 \int_0^1 du_2 \, \frac{\partial f}{\partial u_2} =$$
$$= -\int_0^1 du_1 \big(f(u_1, 1) - f(u_1, 0) \big) = -\int_0^1 f(u_1, 1) du_1 + \int_0^1 f(u_1, 0) du_1 \, .$$

On the other hand, $\int_{\partial B} \omega = \int_0^1 f(u_1, 0) du_1 - \int_0^1 f(u_1, 1) du_1$. □

11h First implications

Here is a counterpart of 11c4.

11h1 Corollary.

$$C_1 \sim C_2 \quad \text{implies} \quad \partial C_1 \sim \partial C_2$$

for arbitrary 2-chains C_1, C_2 in $\mathbb{R}^n$.

Indeed, $\int_{\partial C_1} \omega = \int_{C_1} d\omega = \int_{C_2} d\omega = \int_{\partial C_2} \omega$ for every 1-form ω of class C^1, and therefore also for every 1-form of class C^0, since 11c5 generalizes readily to 1-forms.

Now we return to a question posed in Sect. 10c (after 10c11): is the path function $\gamma \mapsto \int_\gamma \omega$ continuous?

11h2 Proposition. Assume that $\gamma, \gamma_1, \gamma_2, \dots \in C^1\big([t_0, t_1] \to \mathbb{R}^n\big)$, γ_k are bounded in C^1 (that is, $\sup_k \max_t |\gamma_k'(t)| < \infty$), and $\gamma_k \to \gamma$ in C^0 (that is, $\max_t |\gamma_k(t) - \gamma(t)| \to 0$ as $k \to \infty$). Then

$$\int_{\gamma_k} \omega \to \int_\gamma \omega \quad \text{as } k \to \infty$$

for every 1-form ω (of class C^0) on $\mathbb{R}^n$.

11h3 Remark. The condition that γ_k are bounded in C^1 cannot be dropped. Here is a counterexample:

$$\gamma_k(t) = \frac{1}{\sqrt{k}} (\cos kt, \sin kt) \quad \text{for } t \in [0, 2\pi]\,,$$
$$\gamma_k \to \gamma\,, \quad \gamma(t) = (0,0)\,;$$
$$\omega = x\,dy - y\,dx\,;$$
$$\int_{\gamma_k} \omega = \int_0^{2\pi} \frac{1}{k} \big(\cos kt \cdot (\sin kt)' - \sin kt \cdot (\cos kt)' \big)\,\mathrm{d}t = 2\pi \quad \text{for all } k\,;$$
$$\int_\gamma \omega = 0\,.$$

Proof of Prop. 11h2. First, we may assume that ω is of class C^1. Otherwise we approximate it by 1-forms ω_j of class C^1;

$$\omega = \sum_{i=1}^n f_i\,dx_i\,; \quad \omega_j = \sum_{i=1}^n f_{i,j}\,dx_i\,; \quad f_{i,j} \in C^1(\mathbb{R}^n)\,;$$

$f_{i,j} \to f_i$ as $j \to \infty$, uniformly on bounded sets (recall 11c5);

$$\Big| \int_{\gamma_k} \omega - \int_\gamma \omega \Big| \le \Big| \int_{\gamma_k} \omega - \int_{\gamma_k} \omega_j \Big| + \Big| \int_{\gamma_k} \omega_j - \int_\gamma \omega_j \Big| + \Big| \int_\gamma \omega_j - \int_\gamma \omega \Big| ;$$

$$\Big| \int_\gamma \omega_j - \int_\gamma \omega \Big| = \Big| \int_{t_0}^{t_1} \sum_{i=1}^n f_{i,j}\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t - \int_{t_0}^{t_1} \sum_{i=1}^n f_i\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t \Big| \le$$
$$\le \int_{t_0}^{t_1} \sum_{i=1}^n \big| f_{i,j}\big(\gamma(t)\big) - f_i\big(\gamma(t)\big)\big| \cdot |\gamma'(t)|\,\mathrm{d}t \to 0 \quad \text{as } j \to \infty\,;$$

similarly, $\int_{\gamma_k} \omega - \int_{\gamma_k} \omega_j \to 0$ as $j \to \infty$, uniformly in k (since all $\gamma_k(t)$ are a bounded subset of $\mathbb{R}^n$, and all $\gamma_k'(t)$ are bounded). Given $\varepsilon > 0$, we take j such that the first and third terms are less than ε (irrespective of k), and then we take k such that the second term is less than ε.

So, ω is of class C^1. We take $\varepsilon_k \to 0$ such that $|\gamma_k(t) - \gamma(t)| \le \varepsilon_k$ for all t. We introduce boxes $B_k = [t_0, t_1] \times [0, \varepsilon_k] \subset \mathbb{R}^2$ and define singular 2-boxes $\Gamma_k : B_k \to \mathbb{R}^n$ by

$$\Gamma_k(t,u) = \Big(1 - \frac{u}{\varepsilon_k}\Big)\gamma_k(t) + \frac{u}{\varepsilon_k}\gamma(t)\,.$$

We have $\Gamma_k(\cdot, 0) = \gamma_k$ and $\Gamma_k(\cdot, \varepsilon_k) = \gamma$, thus,

$$\partial\Gamma_k = \gamma_k - \gamma + \beta_k - \alpha_k\,,$$

where $\alpha_k, \beta_k : [0, \varepsilon_k] \to \mathbb{R}^n$,

$$\alpha_k(u) = \Big(1 - \frac{u}{\varepsilon_k}\Big)\gamma_k(t_0) + \frac{u}{\varepsilon_k}\gamma(t_0)\,, \quad \beta_k(u) = \Big(1 - \frac{u}{\varepsilon_k}\Big)\gamma_k(t_1) + \frac{u}{\varepsilon_k}\gamma(t_1)\,.$$

We have

$$\int_{\alpha_k} \omega = \int_0^{\varepsilon_k} \sum_{i=1}^n f_i(\alpha_k(u))\alpha_k'(u)\,\mathrm{d}u \to 0 \quad \text{as } k \to \infty\,,$$

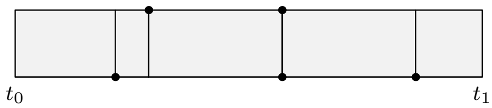
since $\varepsilon_k \to 0$, $|\alpha_k'(u)| = \frac{1}{\varepsilon_k}|\gamma_k(t_0) - \gamma(t_0)| \le 1$, and $f_i(\cdot)$ is bounded. Similarly, $\int_{\beta_k} \omega \to 0$. In order to prove that $\int_{\gamma_k} \omega \to \int_\gamma \omega$ it remains to prove that $\int_{\partial\Gamma_k} \omega \to 0$.

By Theorem 11d3, $\int_{\partial\Gamma_k} \omega = \int_{\Gamma_k} d\omega$. We have $d\omega = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j$ (forget the $f_{i,j}$ used before); by 11e8,

$$\int_{\Gamma_k} d\omega = \int_{B_k} \sum_{i<j} f_{i,j}(x) \frac{\partial(x_i, x_j)}{\partial(t,u)}\,\mathrm{d}t\mathrm{d}u\,,$$

where $x = (x_1, \dots, x_n) = \Gamma_k(t, u)$. In order to prove that $\int_{\Gamma_k} d\omega \to 0$ it remains to check that the integrand is uniformly bounded (since $v(B_k) = (t_1 - t_0)\varepsilon_k \to 0$). We have $\left|\frac{\partial x_i}{\partial t}\right| \le \max\big(|\gamma'_k(t)|, |\gamma'(t)|\big)$ and $\left|\frac{\partial x_i}{\partial u}\right| \le 1$, thus $\frac{\partial(x_i,x_j)}{\partial(t,u)}$ is uniformly bounded. Also $f_{i,j}(x)$ is uniformly bounded (since all $\Gamma_k(t, u)$ are a bounded subset of $\mathbb{R}^n$). □

11h4 Remark. Prop. 11h2 generalizes readily to paths γ_k, γ that are only *piecewise* continuously differentiable. To this end we split B_k as needed,



apply Stokes' theorem to each fragment, and sum up.

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