16 Integration: from single-chart to many-chart

Single-chart pieces of a manifold are combined via partitions of unity. Curvilinear iterated integrals, Stokes' and divergence theorems take their global geometric form.

16a Curvilinear iterated integral

Recall several facts.

* The iterated integral approach (Sect. 7) decomposes an integral over the plane into integrals over parallel lines. It also decomposes an integral over 3-dimensional space into integrals over parallel planes.[1]
* A 3-dimensional integral decomposes into integrals over spheres, see 14b12 and 15d12.
* However, a naive attempt to decompose an integral over the plane into integrals over curves $y = f(x)+a$ fails (see 15d9); a new factor appears, like Jacobian.

Thus, we want to understand, whether or not a 2-dimensional integral decomposes into integrals over curves $\varphi(\cdot) = \text{const}$, and what about a new factor; and what happens in dimension 3 (and more).

First we try dimension $0 + 1$. Let $\varphi \in C^1(\mathbb{R})$, $\forall x \; \varphi'(x) \ne 0$. A set $M_c = \{x : \varphi(x) = c\}$, being a singleton $\{\varphi^{-1}(c)\}$, may be treated as a 0-dimensional manifold in $\mathbb{R}$; naturally, $\int_{M_c} f = f(\varphi^{-1}(c))$. Thus, generally $\int \mathrm{d}c \int_{M_c} f \ne \int_{\mathbb{R}} f$; rather, $\int \mathrm{d}c \int_{M_c} f = \int f(\varphi^{-1}(c))\, \mathrm{d}c = \int f(x)|\varphi'(x)|\, \mathrm{d}x = \int f|\varphi'|$; the new factor $|\varphi'|$ appears. Roughly, it shows how many 0-manifolds M_c appear within an infinitesimal neighborhood of x.

We turn to dimension $1+1$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 near 0, $\varphi(0) = 0$, $(D\varphi)_0 \ne 0$. Then φ is a co-chart of the set $M_0 = \{(x, y) : \varphi(x, y) = 0\}$ around $(0, 0)$, and $\varphi(\cdot)-c$ is a co-chart of $M_c = \{(x, y) : \varphi(x, y) = c\}$ provided that c is small enough. We restrict ourselves to small c and small (x, y), then M_c are 1-manifolds. Assuming that a function $f \in C(\mathbb{R}^2)$ has a compact

[1] Or alternatively, parallel lines. In this course we restrict ourselves to dimension $n+1$; for dimension $n+m$ see the "Coarea formula".

support within the small neighborhood of $(0,0)$, we consider $\int \mathrm{d}c \int_{M_c} f$. It is easy to guess that

$$\int \mathrm{d}c \int_{M_c} f = \int_{\mathbb{R}^2} f|\nabla\varphi| \,, \tag{16a1}$$

since $|\nabla\varphi(x,y)|$ shows roughly, how many curves M_c intersect an infinitesimal neighborhood of (x,y). Note that both sides of (16a1) are invariant under rotations of the plane (since the volume form is well-defined for an n-manifold in an N-dimensional Euclidean space).

The case of a *linear* function φ is simple and instructive. When proving (16a1) for a linear φ we may assume (due to the rotation invariance) that $\varphi(x,y) = ay$. Then $\int_{M_c} f = \int f\bigl(x, \frac{c}{a}\bigr)\,\mathrm{d}x$, $|\nabla\varphi| = |a|$,

$$\int \mathrm{d}c \int_{M_c} f = \int \mathrm{d}c \int \mathrm{d}x\, f\Bigl(x, \frac{c}{a}\Bigr) = a \int \mathrm{d}y \int \mathrm{d}x\, f(x,y) \,,$$

which proves (16a1) for a linear φ. Taking $\varphi(x,y) = ax + by$ with $b \ne 0$ we get

$$M_c = \Bigl\{ \Bigl(x, \frac{c-ax}{b}\Bigr) : x \in \mathbb{R} \Bigr\} \,, \qquad |\nabla\varphi| = \sqrt{a^2+b^2} \,,$$

$$\int_{M_c} f = \int_{\mathbb{R}} f\Bigl(x, \frac{c-ax}{b}\Bigr) \sqrt{1 + \Bigl(-\frac{a}{b}\Bigr)^2}\,\mathrm{d}x \,;$$

$$\int \mathrm{d}c \int_{M_c} f = \frac{\sqrt{a^2+b^2}}{|b|} \iint f\Bigl(x, \frac{c-ax}{b}\Bigr)\,\mathrm{d}x\,\mathrm{d}c \,;$$

$$\int_{\mathbb{R}^2} f|\nabla\varphi| = \sqrt{a^2+b^2} \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \,;$$

(16a1) becomes

$$\frac{1}{|b|} \iint f\Bigl(x, \frac{c-ax}{b}\Bigr)\,\mathrm{d}x\,\mathrm{d}c = \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \,,$$

which follows also from the fact that the Jacobian $\frac{\partial(x,c)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix}$ of the mapping $(x,y) \mapsto (x, ax+by)$ is equal to b.

The former argument (the rotation) fails for nonlinear φ (think, why), but the latter argument (the change of variables) still works, and generalizes to dimension $n+1$, as we'll see soon.

Recall the implicit function theorem 5c1 (for $c = 1$, and some notations changed): if $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}$, $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuously differentiable near (x_0,y_0), $\varphi(x_0,y_0) = 0$, and $\bigl(\frac{\partial\varphi}{\partial y}\bigr)_{(x_0,y_0)} \ne 0$, then there exist open neighborhoods U of x_0 and V of y_0 such that

(a) for every $x \in U$ there exists one and only one $y \in V$ satisfying $\varphi(x,y)=0$;

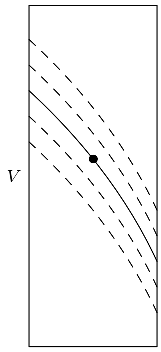
(b) a function $g : U \to V$ defined by $\varphi\big(x, g(x)\big) = 0$ is continuously differentiable, and $\nabla g(x_0) = -\frac{1}{(\frac{\partial \varphi}{\partial y})_{(x_0,y_0)}} \big(\frac{\partial \varphi}{\partial x}\big)_{(x_0,y_0)}$.

Recall also the idea of the proof: a mapping

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \varphi(x,y) \end{pmatrix}$$

is a diffeomorphism $U \times V \to h(U \times V)$, and

$$h^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ g(x) \end{pmatrix}.$$



We need a bit more: there exists an open neighborhood W of 0 in $\mathbb{R}$ such that for every $c \in W$,

(a′) for every $x \in U$ there exists one and only one $y \in V$ satisfying $\varphi(x,y)=c$;

(b′) a function $g_c : U \to V$ defined by $\varphi\big(x, g_c(x)\big) = c$ is continuously differentiable, and $\nabla g_c(x) = -\frac{1}{(\frac{\partial \varphi}{\partial y})_{(x,y)}} \big(\frac{\partial \varphi}{\partial x}\big)_{(x,y)}$ whenever $x \in U$, $y = g_c(x)$.

This is easy to prove; basically, $h^{-1}\begin{pmatrix} x \\ c \end{pmatrix} = \begin{pmatrix} x \\ g_c(x) \end{pmatrix}$; for (b′), differentiate in x the relation $\varphi(x, g_c(x)) = c$.

Thus, for every $c \in W$ the set

$$M_c = \{(x,y) \in U \times V : \varphi(x,y) = c\}$$

is an n-manifold in $\mathbb{R}^{n+1}$; the function $\varphi(\cdot) - c$ is a co-chart of M_c; and the mapping $U \ni x \mapsto \psi_c(x) = \big(x, g_c(x)\big)$ is a chart of the whole M_c; in other words, M_c is the graph of g_c. The set

$$\bigcup_{c \in W} M_c = h^{-1}(U \times W)$$

is an open neighborhood of (x_0, y_0).

16a2 Proposition. For every continuous, compactly supported function f on $\cup_{c \in W} M_c$,

$$\int_W \mathrm{d}c \int_{M_c} f = \int f |\nabla \varphi| \, .$$

16a3 Exercise. Find J_ψ given $\psi(x) = (x, g(x)) \in \mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$ and $g \in C^1(\mathbb{R}^n)$.[1]

Answer: $\sqrt{1 + |\nabla g|^2}$.

Proof of Prop. 16a2. For every $c \in W$,

$$\int_{M_c} f = \int_U f\big(x, g_c(x)\big) \underbrace{\sqrt{1 + |\nabla g_c|^2}}_{J_{\psi_c}} \, \mathrm{d}x$$

due to 16a3; thus, the function $c \mapsto \int_{M_c} f$ is continuous, and

$$\int_W \mathrm{d}c \int_{M_c} f = \iint_{U \times W} f\big(x, g_c(x)\big) \sqrt{1 + |\nabla g_c(x)|^2} \, \mathrm{d}x \, \mathrm{d}c \, .$$

On the other hand, $Dh = \left(\begin{array}{c|c} \mathrm{id} & 0 \\ \hline \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{array} \right)$, therefore $\det(Dh) = \frac{\partial \varphi}{\partial y}$. Also,

$$1 + |\nabla g_c(x)|^2 = 1 + \left(\frac{1}{\big(\frac{\partial \varphi}{\partial y}\big)_{(x,y)}} \right)^2 \left| \Big(\frac{\partial \varphi}{\partial x} \Big)_{(x,y)} \right|^2 = \frac{|\nabla \varphi(x,y)|^2}{\Big(\big(\frac{\partial \varphi}{\partial y}\big)_{(x,y)} \Big)^2}$$

whenever $y = g_c(x)$. Finally, we apply change of variables:

$$\int_W \mathrm{d}c \int_{M_c} f = \iint_{U \times W} f\big(x, g_c(x)\big) \frac{|\nabla \varphi(x, g_c(x))|}{|\det(Dh)_{(x, g_c(x))}|} \, \mathrm{d}x \, \mathrm{d}c =$$

$$= \iint_{U \times W} \frac{f\big(h^{-1}(x,c)\big) |\nabla \varphi(h^{-1}(x,c))|}{|\det(Dh)_{h^{-1}(x,c)}|} \, \mathrm{d}x \, \mathrm{d}c =$$

$$= \iint_{U \times W} f\big(h^{-1}(x,c)\big) |\nabla \varphi(h^{-1}(x,c))| |\det(Dh^{-1})_{(x,c)}| \, \mathrm{d}x \, \mathrm{d}c =$$

$$= \iint_{h^{-1}(U \times W)} f(x,y) |\nabla \varphi(x,y)| \, \mathrm{d}x \, \mathrm{d}y \, .$$

□

16b Many-chart integration

Recall that $\int_{(M,\mathcal{O})} \omega$ is defined by (15c2) whenever $(M, \mathcal{O})$ is an oriented n-manifold and ω a single-chart n-form on M. The linearity,

$$\text{(16b1)} \qquad \int_{(M,\mathcal{O})} (c_1 \omega_1 + c_2 \omega_2) = c_1 \int_{(M,\mathcal{O})} \omega_1 + c_2 \int_{(M,\mathcal{O})} \omega_2 \, ,$$

[1] Hint: in order to avoid working hard on a determinant, use the rotation invariance.

is ensured by (15c2) provided that both forms ω_1, ω_2 have compact supports within the same chart.

The idea of a "partition of unity" was used in Sect. 8h (when proving Th. 8a5) in a rudimentary form: partition into integrable functions. Now we need a bit more: partition into continuous functions.[1]

16b2 Lemma. Let $M \subset \mathbb{R}^N$ be an n-manifold and $K \subset M$ a compact set. Then there exist single-chart continuous functions $f_1, \dots, f_k : M \to [0,1]$ such that $f_1 + \dots + f_k = 1$ on K.

Proof. For every $x \in K$ a function $g_x : y \mapsto \big(\varepsilon_x - |y - x|\big)^+$ is single-chart if ε_x is small enough, continuous, and positive in the open ε_x-neighborhood of x. These neighborhoods are an open covering of K; we choose a finite subcovering and get single-chart functions $g_1, \dots, g_k : M \to [0, \infty)$ whose sum $g = g_1 + \dots + g_k$ is (strictly) positive on K. We take $\varepsilon > 0$ such that $g(\cdot) \ge \varepsilon$ on K and note that functions $f_1, \dots, f_k : M \to [0, \infty)$ defined by

$$f_i(x) = \frac{g_i(x)}{\max(g(x), \varepsilon)}$$

have the required properties. □

It follows that every compactly supported n-form on M is the sum of single-chart n-forms,

$$\omega = \omega_1 + \dots + \omega_k \,, \quad \omega_i = f_i \omega \,.$$

It is tempting to define (assuming that $\mathcal{O}$ is an orientation of M)

$$\int_{(M,\mathcal{O})} \omega = \int_{(M,\mathcal{O})} \omega_1 + \dots + \int_{(M,\mathcal{O})} \omega_k \,; \qquad (16b3)$$

however, does this sum depend on the choice of $\omega_1, \dots, \omega_k$? If $\omega_1 + \dots + \omega_k = \omega = \tilde\omega_1 + \dots + \tilde\omega_{\tilde k}$ then $\omega_1 + \dots + \omega_k + (-\tilde\omega_1) + \dots + (-\tilde\omega_{\tilde k}) = 0$; the question is, whether the corresponding sum of integrals must vanish, or not.

16b4 Lemma. Let $\omega_1, \dots, \omega_\ell$ be single-chart n-forms on an n-manifold M, and $\mathcal{O}$ an orientation of M;

$$\text{if} \quad \omega_1 + \dots + \omega_\ell = 0 \quad \text{then} \quad \int_{(M,\mathcal{O})} \omega_1 + \dots + \int_{(M,\mathcal{O})} \omega_\ell = 0 \,.$$

[1] Still more will be needed in the proof of Th. 16b15: partition into C^1 functions. (Ultimately, partitions into C^∞ functions exist, but we do not need them.)

Proof. Lemma 16b2 gives single-chart continuous functions $f_1, \dots, f_k$ such that $f_1 + \dots + f_k = 1$ on a compact set that supports $\omega_1, \dots, \omega_\ell$. By (16b1), on one hand,

$$\sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} f_i \omega_j = \int_{(M,\mathcal{O})} \underbrace{\sum_{j=1}^{\ell} f_i \omega_j}_{=0} = 0 \, ,$$

since f_i is single-chart; and on the other hand,

$$\sum_{i=1}^{k} \int_{(M,\mathcal{O})} f_i \omega_j = \int_{(M,\mathcal{O})} \underbrace{\sum_{i=1}^{k} f_i \omega_j}_{=\omega_j} = \int_{(M,\mathcal{O})} \omega_j \, ,$$

since ω_j is single-chart. Therefore

$$\sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} \omega_j = \sum_{j=1}^{\ell} \sum_{i=1}^{k} \int_{(M,\mathcal{O})} f_i \omega_j = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} f_i \omega_j = \sum_{i=1}^{k} 0 = 0 \, .$$

□

We see that (16b3) is indeed a correct definition of $\int_{(M,\mathcal{O})} \omega$ whenever ω is a compactly supported n-form on M.

Now we can define the n-dimensional volume of a compact oriented n-manifold $(M, \mathcal{O})$ by

$$V_n(M, \mathcal{O}) = \int_{(M,\mathcal{O})} \mu_{(M,\mathcal{O})} \in (0, \infty)$$

where $\mu_{(M,\mathcal{O})}$ is the volume form on $(M, \mathcal{O})$. However, the Möbius strip should have an area, too!

We want to define

$$\text{(16b5)} \qquad \int_M f = \int_{(G,\psi)} f \mu_{(G,\psi)}$$

for a single-chart $f \in C(M)$; here (G, ψ) is a chart such that f is compactly supported within $\psi(G)$, and $\mu_{(G,\psi)}$ is the volume form on the n-manifold $\psi(G)$ (oriented, even if M is non-orientable). To this end we need a counterpart of Lemma 15c3:

$$\int_{(G_1,\psi_1)} f \mu_{(G_1,\psi_1)} = \int_{(G_2,\psi_2)} f \mu_{(G_2,\psi_2)}$$

whenever (G_1, ψ_1), (G_2, ψ_2) are charts such that $K \subset \psi_1(G_1) \cap \psi_2(G_2)$ for some compact K that supports f. We do it similarly to the proof of 15c3, but this time we split the relatively open set $\tilde{G} = \psi_1(G_1) \cap \psi_2(G_2)$ in two relatively open sets $\tilde{G}_-, \tilde{G}_+$ according to the sign of $\det D\varphi$ (having $\psi_2^{-1} = \varphi \circ \psi_1^{-1}$ on $\tilde{G}$). It remains to take into account that $\mu_{(G_1,\psi_1)} = \mu_{(G_2,\psi_2)}$ on $\tilde{G}_+$ but $\mu_{(G_1,\psi_1)} = -\mu_{(G_2,\psi_2)}$ on $\tilde{G}_-$.

We see that (16b5) is indeed a correct definition of $\int_M f$ for a single-chart $f \in C(M)$. Now, similarly to (16b2), we define

$$\int_M f = \int_M f_1 + \dots + \int_M f_k \tag{16b6}$$

whenever $f = f_1 + \dots + f_k$ with single-chart $f_i \in C(M)$.

16b7 Exercise. (a) Prove that (16b6) is a correct definition of $\int_M f$ for all compactly supported $f \in C(M)$;[1]

(b) formulate and prove linearity and monotonicity of the integral.

Now it is easy to define lower and upper integrals for discontinuous compactly supported functions $M \to \mathbb{R}$ (recall 6i2), and then, Riemann integrability and Jordan measure on an n-manifold in $\mathbb{R}^N$. For functions with no compact support, improper integral may be used. In particular, for a non-compact manifold M,

$$V_n(M) = \sup_{f \le 1} \int_M f = \sup_E V_n(E)$$

where f runs over compactly supported continuous (or just integrable) functions, and E runs over sets Jordan measurable in M. Monotone convergence of volumes (similar to 9c1) holds.

16b8 Exercise. Find the area of the (non-compact) Möbius strip 15b7.

Here is a harder exercise: find the area of the compact non-orientable 2-manifold in $\mathbb{R}^6$ introduced in 15b9.

Curvilinear iterated integral revisited

16b9 Theorem. Let $G \subset \mathbb{R}^n$ be an open set, $n > 1$, $\varphi \in C^1(G)$, $\forall x \in G \;\; \nabla\varphi(x) \ne 0$, and $f \in C(G)$ compactly supported. Then for every $c \in \varphi(G)$ the set $M_c = \{x \in G : \varphi(x) = c\}$ is an $(n-1)$-manifold in $\mathbb{R}^n$, a function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous and compactly supported, and

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} f = \int_G f |\nabla\varphi| \,.$$

[1]Hint: use partitions of unity.

16b10 Remark. A function $c \mapsto V_{n-1}(M_c)$ need not be continuous on $\varphi(G)$. For a counterexample try $G = \{(x, y) : y < g(x)\} \subset \mathbb{R}^2$ and $\varphi(x, y) = y$.

16b11 Exercise. Prove Theorem 16b9.[1]

16b12 Exercise. For $f \in C(\mathbb{R}^n \setminus \{0\})$ prove that

$$\int_{(0,\infty)} \mathrm{d}r \int_{\{x:|x|=r\}} f = \int_{\mathbb{R}^n\setminus\{0\}} f \,,$$

where $\int_{(0,\infty)}$ and $\int_{\mathbb{R}^n\setminus\{0\}}$ are improper; that is, each side of the equality may be a number, $-\infty$, $+\infty$ or $\infty - \infty$.[2]

In particular,

$$\int_{\mathbb{R}^n\setminus\{0\}} f(|x|)\,\mathrm{d}x = \int_{(0,\infty)} V_{n-1}(S_r) f(r)\,\mathrm{d}r \,,$$

where $S_r = \{x : |x| = r\}$ is the sphere. It is easy to see that $V_{n-1}(S_r) = r^{n-1}V_{n-1}(S_1)$; thus,

$$\int_{\mathbb{R}^n\setminus\{0\}} f(|x|)\,\mathrm{d}x = V_{n-1}(S_1) \int_{(0,\infty)} r^{n-1} f(r)\,\mathrm{d}r \,.$$

Now we may take $f(r) = \mathrm{e}^{-r^2}$ and get

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2}\,\mathrm{d}x = \biggl(\int_{\mathbb{R}} \mathrm{e}^{-t^2}\,\mathrm{d}t \biggr)^n = (\sqrt{\pi})^n = \pi^{n/2}$$

(recall 9e); thus,

$$\pi^{n/2} = V_{n-1}(S_1) \int_0^\infty r^{n-1}\mathrm{e}^{-r^2}\,\mathrm{d}r \,.$$

Taking into account that

$$\int_0^\infty r^{n-1}\mathrm{e}^{-r^2}\,\mathrm{d}r = \int_0^\infty t^{(n-1)/2}\mathrm{e}^{-t}\,\frac{\mathrm{d}t}{2\sqrt{t}} = \frac12 \Gamma\Bigl(\frac{n}{2}\Bigr)$$

(recall 9j1), we get[3]

$$V_{n-1}(S_1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \,. \qquad (16b13)$$

[1] Hint: use 16a2 and a partition of unity.
[2] Hint: start with $f \ge 0$, approximate f from below, apply 16b9.
[3] See also Sjamaar, Exer. 9.6.

Alternatively we may use the volume of the ball $B_1 = \{x : |x| < 1\}$,

$$V_n(B_1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} \,,$$

calculated in Sect. 9j.

Divergence theorem revisited

An open set $G \subset \mathbb{R}^n$ is called *regular,* if $(\overline{G})^\circ = G$; that is, the interior of the closure of G is equal to G. (Generally it cannot be less than G, but can be more than G; a simple example: $G = \mathbb{R} \setminus \{0\}$.) Equivalently, G is regular if (and only if) $\partial G = \partial(\mathbb{R}^n \setminus \overline{G})$; that is, the boundary of the exterior of G is equal to the boundary of G.

Let $G \subset \mathbb{R}^n$ be a bounded regular open set, $M \subset \mathbb{R}^n$ a (necessarily compact) $(n-1)$-manifold, and $\partial G = M$ (the topological boundary, nothing "singular"...). We want to prove that the flux of a vector field through M is equal to the integral of its divergence over G. In the language of differential forms (recall 14c8, 14c9) it means a "nonsingular" Stokes' theorem for $k = n-1$: $\int_G d\omega = \int_M \omega$ for every $(n-1)$-form ω on $\mathbb{R}^n$. However, this makes no sense without orienting G and M.

Recall 14c: the hyperface $\{1\} \times [-1,1]^{n-1}$ is a part of the boundary of the cube $(-1,1)^n$; the tangent space to the hyperface is spanned by vectors $e_2, \dots, e_n$; and its orientation conforms to the basis $(e_2, \dots, e_n)$ (in this order), while the orientation of the cube conforms to $(e_1, \dots, e_n)$, of course. And the vector e_1 is the outward unit normal to the hyperface, according to the sign of the inequality $x_1 < 1$ on $(-1,1)^n$.

16b14 Definition. (a) A non-tangent vector $h \in \mathbb{R}^n \setminus T_x M$ is *directed outwards,* if $x - \varepsilon h$ belongs to G and $x + \varepsilon h$ does not belong to G for all ε small enough;

(b) an orientation $\tilde{\mathcal{O}}$ of M conforms at $x \in M$ to an orientation $\mathcal{O}$ of G if $(h_2, \dots, h_n)$ conforms to $\tilde{\mathcal{O}}_x$ whenever h_1 is directed outwards and $(h_1, h_2, \dots, h_n)$ conforms to $\mathcal{O}_x$. (Here $h_2, \dots, h_n \in T_x M$, $h_1 \notin T_x M$.)

For a non-regular G it may happen that $x - \varepsilon h$ and $x + \varepsilon h$ both belong to G (for all ε small enough); but for a regular G either h or $(-h)$ must be directed outwards (and then the other is said to be directed inwards).

16b15 Theorem. Let $G \subset \mathbb{R}^n$ be a bounded regular open set, $M \subset \mathbb{R}^n$ an $(n-1)$-manifold, $\partial G = M$, and orientations $\mathcal{O}$ of G and $\tilde{\mathcal{O}}$ of M conform (at every point of M). Then

$$\int_{(G,\mathcal{O})} d\omega = \int_{(M,\tilde{\mathcal{O}})} \omega$$

for every $(n-1)$-form ω of class C^1 on $\mathbb{R}^n$.

The divergence theorem follows.

16b16 Theorem. Let $G \subset \mathbb{R}^n$ be a bounded regular open set, $M \subset \mathbb{R}^n$ an $(n-1)$-manifold, $\partial G = M$. Then

$$\int_G \operatorname{div} H = \int_M \langle H, \vec{n} \rangle$$

for every vector field H of class C^1 on $\mathbb{R}^n$; here $\vec{n} : M \to \mathbb{R}^n$, $\vec{n}(x)$ is the outward unit normal vector at $x \in M$.

It remains to prove 16b15. Sometimes it is easy to construct an n-chain C such that $C \sim (G, \mathcal{O})$ and $\partial C \sim (M, \tilde{\mathcal{O}})$ in the sense that $\int_C d\omega = \int_{(G,\mathcal{O})} d\omega$ and $\int_{\partial C} \omega = \int_{(M,\mathcal{O})} \omega$; but in general this is problematic. Instead, one turns to a single-chart ω via a partition of unity; and locally M is just the graph of a function.

We restrict ourselves to $n = 2$; the general case is quite similar.

We define a *good box* (for given G and M) as an open box $B \subset \mathbb{R}^2$ such that $M \cap B$ is either the empty set or the graph of a function, either $y = f(x)$ or $x = g(y)$. More exactly, "$y = f(x)$" means here the following: $B = U \times V$ for some open intervals $U, V \subset \mathbb{R}$; $f \in C^1(\overline{U})$, $f(\overline{U}) \subset V$; and $M \cap B = \{(x, f(x)) : x \in U\}$. (The case "$x = g(y)$" is interpreted similarly.)

The closure $G \cup M$ of G is compact, and all good boxes are its open covering. We choose a finite covering: $G \cup M \subset B_1 \cup \dots \cup B_k$, and construct a corresponding partition of unity of class C^1:

$$\begin{gathered} f_1, \dots, f_k : \mathbb{R}^n \to [0,1] \quad \text{are continuously differentiable}, \\ f_i(\cdot) = 0 \quad \text{outside } B_i, \\ f_1 + \dots + f_k = 1 \quad \text{on } G \cup M. \end{gathered}$$

To this end, similarly to the proof of 16b2, we let $g = g_1 + \dots + g_k$, take ε such that $g(\cdot) \ge \varepsilon$ on K, and put

$$f_i(x) = \frac{g_i(x)}{g(x) + \frac{\varepsilon}{2}\bigl(1 - \frac{g(x)}{\varepsilon}\bigr)_+^2};$$

but this time we need $g_i \in C^1$. We obtain g_i by a linear transformation (of arguments) from (say)

$$g(x,y) = h(x)h(y),$$

$$h(t) = \begin{cases} (1-t^2)^2 & \text{for } -1 < t < 1, \\ 0 & \text{otherwise;} \end{cases}$$

then $f_1, \dots, f_k$ have the required properties.

Given an $(n-1)$-form ω of class C^1 on $\mathbb{R}^n$, we have

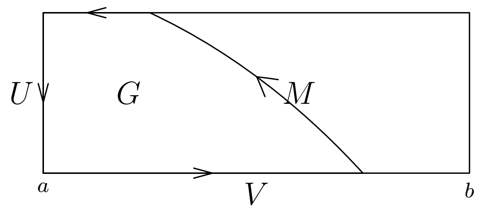
$$\omega = \omega_1 + \dots + \omega_k \quad \text{on } G \cup M \, ,$$

where each $\omega_i = f_i \omega$ is an $(n-1)$-form of class C^1, and $\omega_i = 0$ outside B_i. In order to prove the equality $\int_{(G,\mathcal{O})} d\omega = \int_{(M,\tilde{\mathcal{O}})} \omega$ it is sufficient (due to linearity) to prove the same equality for each ω_i.

The case $M \cap B_i = \emptyset$ is simple: $\int_{(M,\tilde{\mathcal{O}})} \omega_i = 0$ (since $\omega_i = 0$ on M), and $\int_{(G,\mathcal{O})} d\omega = \pm \int_{B_i} d\omega = \pm \int_{\partial B_i} \omega = 0$ (since $\omega_i = 0$ on ∂B_i).

It remains to consider the case "$x = g(y)$" (since the case "$y = f(x)$" is similar).[1] That is, $B_i = V \times U$, $g : \overline{U} \to V$ is continuously differentiable, and $M \cap B_i = \{(g(y), y) : y \in U\}$. We do not know which orientation of B conforms to the given orientation $\mathcal{O}$ of G, but it does not matter, since the other orientation changes the signs of both sides of the equality.

The set $(V \times U) \setminus M$ has exactly two connected components (think, why), one of them being $G \cap (V \times U)$ (think, why). We may assume that $G \cap (V \times U) = \{(x, y) \in V \times U : x < g(y)\}$; in the other case, "$x > g(y)$", we flip the sign of x.



Consider a mapping $\psi_1 : U \to \mathbb{R}^2$, $\psi_1(y) = (g(y), y)$; (U, ψ_1) is a chart of the 1-manifold $M \cap (V \times U)$.

The set $G \cap (V \times U)$ may be treated as a 2-manifold (in $\mathbb{R}^2$); a mapping $\psi_2 : V \times U \to \mathbb{R}^2$,

$$\psi_2(x, y) = \Big(a + \frac{x-a}{b-a}(g(y) - a), y \Big) \, ,$$

where $(a, b) = V$, is a diffeomorphism $V \times U \to G \cap (V \times U)$; and $(V \times U, \psi_2)$ is a chart of $G \cap (V \times U)$.

These charts, (U, ψ_1) and $(V \times U, \psi_2)$, lead to orientations, $\mathcal{O}_1$ on $M \cap (V \times U)$ and $\mathcal{O}_2$ on $G \cap (V \times U)$, and these two orientations conform (according to 16b14(b)) at every $(g(y), y) \in M \cap (V \times U)$; here is why. The vector $(g'(y), 1) \in T_{(g(y),y)} M$ conforms to $\mathcal{O}_1$; the vector $(1, 0)$ is directed

[1] Why prefer "$x = g(y)$" to "$y = f(x)$"? Since our preferred hyperface $\{1\} \times [-1, 1]^{n-1}$ of $[-1, 1]^n$ for $n = 2$ is "$x = 1$", not "$y = \dots$".

outwards; and the basis $\big((1,0),(g'(y),1)\big)$ conforms to $\mathcal{O}_2$, since $\left|\begin{smallmatrix}1 & 0\\ g'(y) & 1\end{smallmatrix}\right| > 0$, and $\det D\psi_2 > 0$ as well.

We apply Stokes' theorem to the singular box $\Gamma : \overline{V} \times \overline{U} \to \mathbb{R}^2$, $\Gamma(x,y) = \big(a + \frac{x-a}{b-a}(g(y)-a), y\big)$, getting $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$. It remains to note that

$$\int_\Gamma d\omega = \int_{(G\cap(V\times U),\mathcal{O}_2)} d\omega\,, \quad \int_{\partial\Gamma} \omega = \int_{(M\cap(V\times U),\mathcal{O}_1)} \omega\,.$$

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