7 Iterated integral

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Iterated integral is an indispensable tool for calculating multidimensional integrals (in particular, volumes). It also leads to a result about integrals (including one-dimensional) that depend on a parameter.

7a What is the problem

According to 6k10,

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \to \int_{\mathbb{R}^2} f \quad \text{as } \varepsilon \to 0$$

for every integrable $f : \mathbb{R}^2 \to \mathbb{R}$. The double summation is evidently equivalent to iterated summation,

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) = \varepsilon \sum_{k\in\mathbb{Z}} \biggl(\varepsilon \sum_{l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \biggr) ,$$

which suggests that

$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y \biggr) \mathrm{d}x \, ,$$

(alternative notation: $\iint f(x,y)\,\mathrm{d}x\mathrm{d}y = \int \mathrm{d}x \int \mathrm{d}y\, f(x,y)$, and the like), that is,

$$\text{(7a1)} \qquad \int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(x \mapsto \int_{\mathbb{R}} f_x \biggr) ,$$

where $f_x : \mathbb{R} \to \mathbb{R}$ (denoted also $f(x,\cdot)$) is defined by

$$f_x(y) = f(x,y) \, .$$

It should be very useful, to integrate with respect to one variable at a time.

Related problems:

* does integrability of f imply integrability of f_x for every x?
* is the function $x \mapsto \int_{\mathbb{R}} f_x$ integrable?
* is the two-dimensional integral equal to the iterated integral?
* if the iterated integral is well-defined, does it follow that f is integrable?

And, of course, we need a multidimensional theory; $\mathbb{R}^2$ is only the simplest case.

7b Lipschitz functions

Recall 6g8: a continuous function on a box is integrable.

7b1 Proposition. Let $f : B \to \mathbb{R}$ be a Lipschitz function on a box $B = I_1 \times I_2 \subset \mathbb{R}^2$. Then
(a) for every $x \in I_1$ the function f_x is Lipschitz continuous on I_2;
(b) the function $x \mapsto \int_{I_2} f_x$ is Lipschitz continuous on I_1;

(c) $$\int_B f = \int_{I_1} \Big(x \mapsto \int_{I_2} f_x \Big) \, .$$

It is given that f is L-Lipschitz for some $L \in (0,\infty)$. We reduce the general case to the case $L = 1$ by turning to the function $\frac{1}{L} f$.

We reduce the general box B of the form $[s_1,t_1] \times [s_2,t_2]$ to a box of the form $[0,t_1] \times [0,t_2]$ by translation, according to 6c7. Further, we reduce it to the square $[0,1] \times [0,1]$ by rescaling, according to 6d18. That is, we introduce a Lipschitz function $g : [0,1] \times [0,1] \to \mathbb{R}$ by $g(x,y) = f(t_1 x, t_2 y)$; by 6d18,

$$t_1 t_2 \int_{[0,1]\times[0,1]} g = \int_B f \, .$$

We note that $g_x(y) = f_{t_1 x}(t_2 y)$; Lipschitz continuity of g_x implies Lipschitz continuity of $f_{t_1 x}$, and $t_2 \int_{[0,1]} g_x = \int_{[0,t_2]} f_{t_1 x}$ (by 6d18 again). Further, Lipschitz continuity of $x \mapsto \int_{[0,1]} g_x$ implies Lipschitz continuity of $x \mapsto \int_{[0,t_2]} f_x$, and

$$t_1 \int_{[0,1]} \Big(x \mapsto \int_{[0,t_2]} f_{t_1 x} \Big) = \int_{[0,t_1]} \Big(x \mapsto \int_{[0,t_2]} f_x \Big)$$

(by 6d18 once again), that is,

$$t_1 t_2 \int_{[0,1]} \Big(x \mapsto \int_{[0,1]} g_x \Big) = \int_{[0,t_1]} \Big(x \mapsto \int_{[0,t_2]} f_x \Big) \, .$$

Now the equality (7b1)(c) for g implies the same for f.

We need a quantitative version of 6k10.

7b2 Lemma. For every 1-Lipschitz function $f : [0,1]^n \to \mathbb{R}$ and every $K = 1, 2, \dots$

$$\bigg| \frac{1}{K^n} \sum_{1 \le k_1, \dots, k_n \le K} f\Big(\frac{k_1 - 0.5}{K}, \dots, \frac{k_n - 0.5}{K} \Big) - \int_{[0,1]^n} f \bigg| \le \frac{\sqrt{n}}{2K} \,.$$

Proof. Consider a partition P of $[0,1]^n$ into K^n δ-pixels (as defined before 6k11) with $\delta = 1/K$. Every point of a pixel C is $\frac{1}{2}\delta\sqrt{n}$-close to the center $(\frac{k_1-0.5}{K}, \dots, \frac{k_n-0.5}{K})$ of the pixel (as noted in the proof of 6m2); the Lipschitz continuity gives

$$f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) - \frac{1}{2}\delta\sqrt{n} \le \inf_C f \le \sup_C f \le f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) + \frac{1}{2}\delta\sqrt{n} \,;$$

$$\sum_{1 \le k_1, \dots, k_n \le K} \delta^n \Big(f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) - \frac{1}{2}\delta\sqrt{n} \Big) \le L(f,P) \le \int_{[0,1]^n} f \le$$

$$\le U(f,P) \le \sum_{1 \le k_1, \dots, k_n \le K} \delta^n \Big(f\big(\tfrac{k_1-0.5}{K}, \dots, \tfrac{k_n-0.5}{K}\big) + \frac{1}{2}\delta\sqrt{n} \Big) \,.$$

□

Proof of Prop. 7b1 for a 1-Lipschitz function f on $B = [0,1] \times [0,1]$.

(a) $|f_x(y_1) - f_x(y_2)| = |f(x,y_1) - f(x,y_2)| \le |(0, y_1 - y_2)| = |y_1 - y_2|$, thus f_x is 1-Lipschitz.

(b) $|(f_{x_1} - f_{x_2})(y)| = |f(x_1,y) - f(x_2,y)| \le |(x_1 - x_2, 0)| = |x_1 - x_2|$, therefore $|\int_{[0,1]} f_{x_1} - \int_{[0,1]} f_{x_2}| \le |x_1 - x_2|$, which shows that the function $x \mapsto \int_{[0,1]} f_x$ is 1-Lipschitz.

(c) Lemma 7b2 applied to f (and $n = 2$) gives for arbitrary $M = 1, 2, \dots$

$$\bigg| \frac{1}{M^2} \sum_{k,l=1}^{M} f\big(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M}\big) - \int_B f \bigg| \le \frac{\sqrt{2}}{2M} \,.$$

The same lemma applied to f_x (and $n = 1$) gives for every x

$$\bigg| \frac{1}{M} \sum_{l=1}^{M} f_x\big(\tfrac{l-0.5}{M}\big) - \int_{[0,1]} f_x \bigg| \le \frac{1}{2M} \,.$$

The same lemma (again!) applied to the function $x \mapsto \int_{[0,1]} f_x$ (and $n = 1$) gives

$$\bigg| \frac{1}{M} \sum_{k=1}^{M} \int_{[0,1]} f_{\frac{k-0.5}{M}} - \int_{[0,1]} \Big(x \mapsto \int_{[0,1]} f_x \Big) \bigg| \le \frac{1}{2M} \,.$$

Thus,

$$\left| \int_B f - \int_{[0,1]} \Big(x \mapsto \int_{[0,1]} f_x \Big) \right| \le$$

$$\le \left| \frac{1}{M^2} \sum_{k,l=1}^{M} f\big(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M}\big) - \frac{1}{M} \sum_{k=1}^{M} \int_{[0,1]} f_{\frac{k-0.5}{M}} \right| + \frac{\sqrt{2}}{2M} + \frac{1}{2M}$$

and

$$\left| \frac{1}{M^2} \sum_{k,l=1}^{M} f\big(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M}\big) - \frac{1}{M} \sum_{k=1}^{M} \int_{[0,1]} f_{\frac{k-0.5}{M}} \right| =$$

$$= \left| \frac{1}{M} \sum_{k=1}^{M} \Big(\frac{1}{M} \sum_{l=1}^{M} f\big(\tfrac{k-0.5}{M}, \tfrac{l-0.5}{M}\big) - \int_{[0,1]} f_{\frac{k-0.5}{M}} \right| \le$$

$$\le \frac{1}{M} \sum_{k=1}^{M} \frac{1}{2M} = \frac{1}{2M} \,.$$

Finally,

$$\left| \int_B f - \int_{[0,1]} \Big(x \mapsto \int_{[0,1]} f_x \Big) \right| \le \frac{\sqrt{2}+2}{2M}$$

for all M. □

Here is a straightforward generalization of Prop. 7b1.

7b3 Proposition. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and a Lipschitz function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then
(a) for every $x \in B_1$ the function f_x is Lipschitz continuous on B_2;
(b) the function $x \mapsto \int_{B_2} f_x$ is Lipschitz continuous on B_1;
(c)
$$\int_B f = \int_{B_1} \Big(x \mapsto \int_{B_2} f_x \Big) \,.$$

7b4 Exercise. Prove Prop. 7b3.

Similarly, for a Lipschitz function $f : B_1 \times B_2 \to \mathbb{R}$,

$$\int_B f = \int_{B_2} \Big(y \mapsto \int_{B_1} f^y \Big)$$

where $f^y(x) = f(x,y)$. This claim reduces to Prop. 7b3 taking $\tilde{f}(y,x) = f(x,y)$. Ultimately,

$$\int \mathrm{d}x \int \mathrm{d}y \, f(x,y) = \iint f(x,y) \, \mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x \, f(x,y) \,.$$

That is, the two iterated integrals are equal to the "non-iterated" ("double"? "single"?) integral (and therefore equal to each other).

7b5 Exercise. Prove that

$$\int_{B_1\times B_2} f(x_1,\dots,x_m)g(y_1,\dots,y_n)\,\mathrm{d}x_1\dots\mathrm{d}x_m\,\mathrm{d}y_1\dots\mathrm{d}y_n = \\ = \Big(\int_{B_1} f(x_1,\dots,x_m)\,\mathrm{d}x_1\dots\mathrm{d}x_m\Big)\Big(\int_{B_2} g(y_1,\dots,y_n)\,\mathrm{d}y_1\dots\mathrm{d}y_n\Big)$$

for Lipschitz functions $f : B_1 \to \mathbb{R}$, $g : B_2 \to \mathbb{R}$.

7b6 Exercise. Calculate each integral in two ways:
(a) $\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{x+y}$;
(b) $\int_0^1 \mathrm{d}y \int_0^{\pi/2} \mathrm{d}x\, xy\cos(x+y)$.

7b7 Exercise. Calculate integrals
(a) $\int_{[0,1]^n}(x_1^2+\dots+x_n^2)\,\mathrm{d}x_1\dots\mathrm{d}x_n$;
(b) $\int_{[0,1]^n}(x_1+\dots+x_n)^2\,\mathrm{d}x_1\dots\mathrm{d}x_n$.

7c Some counterexamples

7c1 Example. Integrability of f does not imply integrability of f_x for every x.

Define $f : [0,1]\times[0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(\cdot,\cdot) = 0$ outside a set $\{1/2\}\times[0,1]$ of area 0, therefore f is integrable (recall 6g). However, $f_{1/2}$ is not integrable (recall 6b2).

7c2 Example. Existence of the iterated integral[1] does not imply boundedness (the more so, integrability) of f, even if f is positive and symmetric in the sense that $f(x,y) = f(y,x)$ (and therefore the iterated integrals $\int \mathrm{d}x \int \mathrm{d}y\, f(x,y)$, $\int \mathrm{d}y \int \mathrm{d}x\, f(x,y)$ are both well-defined, and equal).

Define $f : [0,1]\times[0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{x+y}} & \text{if } x/2 < y < 2x, \\ 0 & \text{otherwise} \end{cases}$$

[1]That is, integrability of f_x for all x *and* integrability of the function $x \mapsto \int f_x$.

and observe that

$$\int_{[0,1]} f_x = \int_{x/2}^{2x} \frac{\mathrm{d}y}{\sqrt{x+y}} = 2\sqrt{x+y}\Big|_{y=x/2}^{y=2x} = 2\sqrt{3x} - 2\sqrt{3x/2} = \text{const} \cdot \sqrt{x}$$

for $x \le 1/2$, and $\int_{x/2}^{1} \frac{\mathrm{d}y}{\sqrt{x+y}} = 2\sqrt{x+1} - 2\sqrt{3x/2}$ for $x \ge 1/2$.

7c3 Example. Existence of both iterated integrals does not imply their equality, even if f is antisymmetric in the sense that $f(x,y) = -f(y,x)$.

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x-y}{(x+y)^3} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise;} \end{cases}$$

observe that each f_x is continuously differentiable (therefore Lipschitz), and

$$\int_{[0,1]} f_x = \int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y = \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,\mathrm{d}y =$$
$$= 2x \int_0^1 \frac{\mathrm{d}y}{(x+y)^3} - \int_0^1 \frac{\mathrm{d}y}{(x+y)^2} = 2x \cdot \Big(-\frac{1}{2}\Big) \frac{1}{(x+y)^2}\Big|_{y=0}^{y=1} - (-1) \cdot \frac{1}{x+y}\Big|_{y=0}^{y=1} =$$
$$= -x\Big(\frac{1}{(x+1)^2} - \frac{1}{x^2}\Big) + \Big(\frac{1}{x+1} - \frac{1}{x}\Big) = \frac{-x+(x+1)}{(x+1)^2} = \frac{1}{(x+1)^2},$$

a positive, continuously differentiable function on $[0,1]$. Its integral is positive (in fact, $1/2$). By the antisymmetry, the other iterated integral is negative (in fact, $-1/2$).

7c4 Example. Existence of the iterated integral does not imply integrability of f even if f is *bounded* and symmetric.

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by[1]

$$f(x,y) = \begin{cases} 1 & \text{if } x\sqrt{2}+y \text{ and } x+y\sqrt{2} \text{ are (both) rational,} \\ 0 & \text{otherwise.} \end{cases}$$

If $f(x,y_1) = f(x,y_2) = 1$ then $y_1 - y_2 = (x\sqrt{2}+y_1) - (x\sqrt{2}+y_2)$ is rational and $(y_1-y_2)\sqrt{2} = (x+y_1\sqrt{2}) - (x+y_2\sqrt{2})$ is rational, therefore $y_1 = y_2$. It means that each $f_x(\cdot) = 0$ outside at most one point. Similarly, each f^y vanishes outside at most one point. Thus, $\int f_x = 0$ for all x, and $\int f^y = 0$ for all y. Nevertheless f is not integrable, since it equals 1 on a dense countable set of points of the form $(q\sqrt{2}-r, r\sqrt{2}-q)$ with rational q, r; and f vanishes on the (dense) complement of this countable set.

[1] Alternatively, $f(x,y) = 1$ whenever $(x,y) = \big((2k-1)/2^n, (2l-1)/2^n\big)$.

7c5 Remark. One may wonder, does existence of both iterated integrals imply their equality if f is just bounded (but not Lipschitz, nor integrable)? The answer is affirmative.[1] Try to prove it yourself if you are ambitious enough, but be warned that you'll probably need something not learned yet in this course.

7d Integrable functions

7d1 Theorem. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and an integrable function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then the iterated integrals

$$\int_{B_1} \mathrm{d}x \, {}_*\!\!\int_{B_2} \mathrm{d}y \, f(x,y) \, , \qquad \int_{B_1} \mathrm{d}x \, {}^*\!\!\int_{B_2} \mathrm{d}y \, f(x,y) \, ,$$

$$\int_{B_2} \mathrm{d}y \, {}_*\!\!\int_{B_1} \mathrm{d}x \, f(x,y) \, , \qquad \int_{B_2} \mathrm{d}y \, {}^*\!\!\int_{B_1} \mathrm{d}x \, f(x,y)$$

are well-defined and equal to

$$\iint_B f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

Clarification. The claim that $\int \mathrm{d}x \, {}_*\!\int \mathrm{d}y \, f(x,y)$ is well-defined means that the function $x \mapsto {}_*\!\int \mathrm{d}y \, f(x,y)$ is integrable.

The equality

$$\int \Big(x \mapsto {}_*\!\!\int f_x \Big) = \int \Big(x \mapsto {}^*\!\!\int f_x \Big)$$

implies integrability (with the same integral) of every function sandwiched between the lower and upper integrals. It is convenient to interpret $x \mapsto \int f_x$ as *any* such function and write, as before,

$$\int_B f = \int_{B_1} \Big(x \mapsto \int_{B_2} f_x \Big)$$

and

$$\int \mathrm{d}x \int \mathrm{d}y \, f(x,y) = \iint f(x,y) \, \mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x \, f(x,y)$$

even though f_x may be non-integrable for some x.

Theorem 7d1 is proved via sandwiching, — either by step functions (recall Sect. 6g) or Lipschitz functions (recall Sect. 6i). Let us use the latter.

[1] In Riemann integration, of course. In Lebesgue integration the corresponding problem is much harder.

Proof. By Prop. 6i2, ${}^*\!\int_B f = \inf_{g \ge f} \int_B g$ where g runs over all Lipschitz functions. For every such g, $\int_B g = \int_{B_1} (x \mapsto \int_{B_2} g_x)$ by Prop. 7b1. We have $\int_{B_2} g_x = {}^*\!\int_{B_2} g_x \ge {}^*\!\int_{B_2} f_x$ (since $g_x \ge f_x$), thus, $\int_B g \ge {}^*\!\int_{B_1} (x \mapsto {}^*\!\int_{B_2} f_x)$ for all these g. Therefore

$$ {}^*\!\!\int_B f \ge {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

Similarly (or via $(-f)$),

$$ {}_*\!\!\int_B f \le {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) . $$

Using integrability of f,

$$ \int_B f \le {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) \le \int_B f , $$

therefore

$$ \int_B f = {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) = {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

Integrability of the function $x \mapsto {}_*\!\int_{B_2} f_x$ follows, since

$$ \int_B f = {}_*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) \le {}^*\!\!\int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) = \int_B f . $$

Similarly, the function $x \mapsto {}^*\!\int_{B_2} f_x$ is also integrable. Thus,

$$ \int_B f = \int_{B_1} \Big(x \mapsto {}_*\!\!\int_{B_2} f_x \Big) = \int_{B_1} \Big(x \mapsto {}^*\!\!\int_{B_2} f_x \Big) . $$

The other two iterated integrals are treated similarly (or via $\tilde f(y,x) = f(x,y)$). □

7d2 Exercise. Give another proof of 7d1, via sandwiching by step functions.[1]

7d3 Exercise. Generalize 7b5 to integrable functions

(a) assuming integrability of the function $(x,y) \mapsto f(x)g(y)$,

(b) deducing integrability of the function $(x,y) \mapsto f(x)g(y)$ from integrability of f and g (via sandwich).

[1]Hint: first, consider $f = 1\!\!1_C$ for a box $C \subset B$.

7d4 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ are Jordan measurable sets then the set $E = E_1 \times E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

Prove it.

7d5 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^{m+n}$ are Jordan measurable sets then the set $E = \{(x,y) \in E_2 : x \in E_1\} = (E_1 \times \mathbb{R}^n) \cap E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

Prove it.

Applying Theorem 7d1 to a function $f1\!\!1_E$ and taking 6j5 into account we get the following.

7d6 Corollary. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be integrable on every box, and $E \subset \mathbb{R}^{m+n}$ a Jordan measurable set; then

$$\int_E f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{E_x} f_x \Big)$$

where $E_x = \{y : (x,y) \in E\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

Clarification. First, note that $\{x : E_x \ne \emptyset\}$ is bounded, and $\int_\emptyset f_x = 0$. Second: it may happen that $\int_{E_x} f_x$ is ill-defined for some x; then it is interpreted as anything between ${}_*\!\int f_x 1\!\!1_{E_x}$ and ${}^*\!\int f_x 1\!\!1_{E_x}$.

In particular, taking $f(\cdot) = 1$ we get

$$v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x)\, \mathrm{d}x$$

where v_k is the Jordan measure in $\mathbb{R}^k$. For instance, the volume of a 3-dimensional geometric body is the 1-dimensional integral of the area of the 2-dimensional section of the body.

7d7 Corollary. If Jordan measurable sets $E, F \subset \mathbb{R}^3$ satisfy $v_2(E_x) = v_2(F_x)$ for all x then $v_3(E) = v_3(F)$.[1]

This is a modern formulation of the Cavalieri's principle:[2] [3]

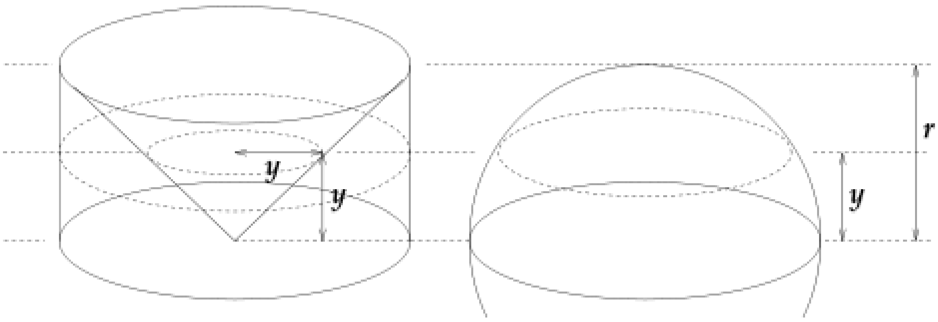
Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

[1]It is sufficient to check the equality for all x of a dense subset of $\mathbb{R}$ (since two *Riemann integrable* functions equal on a dense set must have equal integrals).

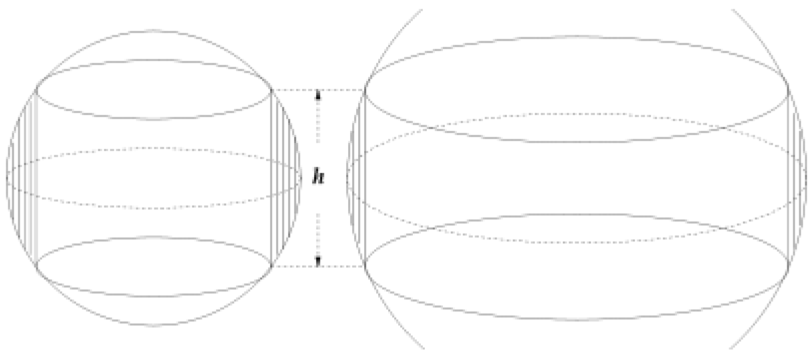
[2]Bonaventura Francesco Cavalieri (in Latin, Cavalerius) (1598–1647), Italian mathematician.

[3]Images (and some text) from Wikipedia, "Cavalieri's principle".

Before emergence of the integral calculus, Cavalieri was able to calculate some volumes by ingenious use of this principle. Here are two examples. First, the volume of the upper half of a sphere is equal to the volume of a cylinder minus volume of a cone:



Second, when a hole of length h is drilled straight through the center of a sphere, the volume of the remaining material surprisingly does not depend on the size of the sphere:



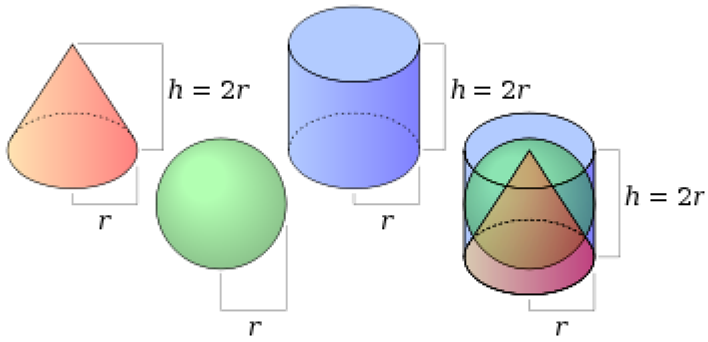
7d8 Exercise. Check the two results of Cavalieri noted above.

7d9 Exercise. Check a famous result of Archimedes:[1] [2] a sphere inscribed

[1] Archimedes ($\approx$ 287–212 BC), a Greek mathematician, generally considered to be the greatest mathematician of antiquity and one of the greatest of all time.
Cicero describes visiting the tomb of Archimedes, which was surmounted by a sphere inscribed within a cylinder. Archimedes ... regarded this as the greatest of his mathematical achievements.

[2] Images (and some text) from Wikipedia, "Volume" (section "Volume ratios for a cone, sphere and cylinder of the same radius and height").

within a cylinder has two thirds of the volume of the cylinder.



Moreover, show that the volumes of a cone, sphere and cylinder of the same radius and height are in the ratio 1 : 2 : 3.

Another important special case of 7d6:

$$E = \{(x,t) : x \in B,\, g(x) \le t \le h(x)\} \subset \mathbb{R}^{n+1}$$

where $B \subset \mathbb{R}^n$ is a box and $g.h : B \to \mathbb{R}$ integrable functions satisfying $g \le h$ (recall Sect. 6h). In this case $E_x = [g(x), h(x)]$, and we get

$$\int_E f = \int_B \Big(x \mapsto \int_{[g(x),h(x)]} f_x \Big) = \int_B \mathrm{d}x \int_{g(x)}^{h(x)} \mathrm{d}t\, f(x,t)\,.$$

Applying this to $f(x,t)\mathbb{1}_F(x)$ (in place of $f(x,t)$) for a Jordan measurable set $F \subset \mathbb{R}^n$ we get

$$\int_E f = \int_F \mathrm{d}x \int_{g(x)}^{h(x)} \mathrm{d}t\, f(x,t)$$

where $F = \{x \in B : g(x) \le t \le h(x)\}$ (assuming that this set is Jordan measurable).

7d10 Exercise. Calculate the integral

$$\iiint_E (x_1^2 + x_2^2 + x_3^2)\, dx_1 dx_2 dx_3\,,$$

where E is the simplex in $\mathbb{R}^3$ bounded by the planes $\{x_1 + x_2 + x_3 = a\}$, $\{x_i = 0\}$, $1 \le i \le 3$.

Answer: $a^5/20$.

7d11 Exercise. Find the volume of the intersection of two solid cylinders in $\mathbb{R}^3$: $\{x_1^2 + x_2^2 \le 1\}$ and $\{x_1^2 + x_3^2 \le 1\}$.

Answer: $16/3$.

7d12 Exercise. Find the volume of the solid in $\mathbb{R}^3$ under the paraboloid $\{x_1^2 + x_2^2 - x_3 = 0\}$ and above the square $[0,1]^2 \times \{0\}$.

Answer: 2/3.

7d13 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n \, f(x_n) = \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!} \, dt \, .$$

7d14 Example. Let us calculate the integral

$$\int_{[0,1]^n} \max(x_1, \dots, x_n) \, dx_1 \dots dx_n \, .$$

First of all, by symmetry, we assume that $1 \ge x_1 \ge x_2 \ge \dots \ge x_n \ge 0$, and multiply the answer by $n!$. Then $\max(x_1, ..., x_n) = x_1$, and we get

$$n! \int_0^1 x_1 \, dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n = n! \int_0^1 \frac{x_1^n \, dx_1}{(n-1)!} = \frac{n}{n+1} \, .$$

7d15 Exercise. Compute the integral $\int_{[0,1]^n} \min(x_1, \dots, x_n) \, dx_1 \dots dx_n$.

Answer: $\frac{1}{n+1}$.

7d16 Exercise. Find the volume of the n-dimensional simplex

$$\{x : x_1, ..., x_n \ge 0, \ x_1 + ... + x_n \le 1\} \, .$$

Answer: $\frac{1}{n!}$.

7d17 Exercise. Suppose the function f depends only on the first coordinate. Then

$$\int_{\mathbb{B}} f(x_1) \, dx = v_{n-1} \int_{-1}^1 f(x_1)(1 - x_1^2)^{(n-1)/2} \, dx_1 \, ,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$, and v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

The next exercises examine further a very interesting phenomenon of “concentration of high-dimensional volume” touched before, in 6h4(b); it was seen there that in high dimension the volume of a ball concentrates near the sphere, and now we'll see that it also concentrates near a hyperplane![1]

7d18 Exercise. Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$, and $P = \{x \in \mathbb{B} : |x_1| < 0.01\}$. What is larger, $v_n(P)$ or $v_n(\mathbb{B} \setminus P)$, if n is sufficiently large?

[1] Do you see a contradiction in these claims?

7d19 Exercise. Given $\varepsilon > 0$, show that the quotient

$$\frac{v_n(\{x \in \mathbb{B} : |x_1| > \varepsilon\})}{v_n(\mathbb{B})}$$

tends to zero as $n \to \infty$.[1]

7d20 Exercise. * Find the asymptotic behavior of the quotient above as $n \to \infty$.

7e Differentiation under the integral sign

Integration of the function $x \mapsto \int f_x$ is useful, but differentiation of this function is also widely used. Imagine for instance that a function depends on time: $f_t(x) = f(t, x)$. Then its integral depends on time, too: $t \mapsto \int f(t, x)\,\mathrm{d}x$. According to the so-called Leibniz integral rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f(t, x)\,\mathrm{d}x = \int \Big(\frac{\partial}{\partial t} f(t, x) \Big) \mathrm{d}x$$

under appropriate conditions.[2]

Instead of differentiating $\int f(t, x)\,\mathrm{d}x$ we'll integrate $\int \big(\frac{\partial}{\partial t} f(t, x) \big) \mathrm{d}x$; this little trick shifts the work onto the iterated integral theorem!

7e1 Theorem. Let $B \subset \mathbb{R}^n$ be a box, and $f, g : B \times [0, 1] \to \mathbb{R}$ Lipschitz functions such that $f'_x(t) = g_x(t)$ for all $x \in B$, $t \in (0, 1)$. Then $F'(t) = G(t)$ for all $t \in (0, 1)$, where $F(t) = \int_B f(x, t)\,\mathrm{d}x$ and $G(t) = \int_B g(x, t)\,\mathrm{d}x$.

Clarification. By "$F'(t) = G(t)$" we mean that the derivative exists and equals $G(t)$; and "$f'_x(t) = g_x(t)$" is interpreted similarly.

Proof. We know (recall Sect. 7b) that F and G are Lipschitz continuous. It is sufficient to prove that $\int_0^t G(s)\,\mathrm{d}s = F(t) - F(0)$ for all $t \in (0, 1)$. We have $f_x(t) - f_x(0) = \int_0^t g_x(s)\,\mathrm{d}s$, therefore

$$\begin{aligned} F(t) - F(0) &= \int_B \big(f(x, t) - f(x, 0) \big)\,\mathrm{d}x = \int_B \mathrm{d}x \int_0^t \mathrm{d}s\, g(x, s) = \\ &= \int_0^t \mathrm{d}s \int_B \mathrm{d}x\, g(x, s) = \int_0^t \mathrm{d}s\, G(s)\,. \end{aligned}$$

□

[1] Hint: the quotient equals $\frac{\int_\varepsilon^1 (1-t^2)^{(n-1)/2}\,dt}{\int_0^1 (1-t^2)^{(n-1)/2}\,dt}$.

[2] The conditions of Th. 7e1 can be relaxed in several aspects; but I prefer to keep the proof short.