9 Convergence of volumes and integrals

Jordan measure and Riemann integral are generalized to unbounded sets and functions via limiting procedures.

9a What is the problem

The n-dimensional unit ball in the l_p metric,

$$E = \{(x_1, \dots, x_n) : |x_1|^p + \dots + |x_n|^p \le 1\},$$

is a Jordan measurable set, and its volume is a Riemann integral,

$$v(E) = \int_{\mathbb{R}^n} 1\!\!1_E,$$

of a bounded function with bounded support. In Sect. 9j we'll calculate it:

$$v(E) = \frac{2^n \Gamma^n\big(\frac{1}{p}\big)}{p^n \Gamma\big(\frac{n}{p} + 1\big)}$$

where Γ is a function defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t \quad \text{for } s > 0;$$

here the integrand has no bounded support; and for $s = \frac{1}{p} < 1$ it is also unbounded (near 0). Thus we need a more general, so-called improper integral, even for calculating the volume of a bounded body!

In relatively simple cases the improper integral may be treated via *ad hoc* limiting procedure adapted to the given function; for example,

$$\int_0^\infty t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t = \lim_k \int_{1/k}^k t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t \, .$$

In more complicated cases it is better to have a theory able to integrate rather general functions on rather general n-dimensional sets. Different functions may tend to infinity on different subsets (points, lines, surfaces), and still, we expect $\int (f+g) = \int f + \int g$ (linearity) to hold, as well as change of variables, iterated integral etc.[1]

9b Improper Jordan measure

9b1 Lemma. $v_*(A) = \sup\{v(E) : \text{Jordan } E \subset A\}$ for all bounded $A \subset \mathbb{R}^n$.

Proof. Clearly, $v_*(A) \ge \sup_E v(E)$; we have to prove that $v_*(A) \le \sup_E v(E)$. We have

$$v_*(A) \overset{6f1}{=} {}_*\!\int_{\mathbb{R}^n} 1\!\!1_A \overset{6e4}{=} {}_*\!\int_B 1\!\!1_A \overset{(6g7)}{=} \sup_{h \le 1\!\!1_A} \int_B h$$

where h runs over step functions on B. The set $E = \{x \in B : h(x) > 0\} \subset A$ is Jordan (just a finite union of boxes), and $\int_B h \le v(E)$ (since $h \le 1\!\!1_E$), thus $v_*(A) \le \sup_E v(E)$. □

We extend the inner Jordan measure v_* (defined in 6f1 for bounded sets) to unbounded sets $X \subset \mathbb{R}^n$:

(9b2) $$v_*(X) = \sup\{v(E) : \text{Jordan } E \subset X\} \in [0, \infty] \, .$$

9b3 Exercise.

$$v_*(X) = \lim_{r \to \infty} v_*(X_r) \quad \text{for all } X \subset \mathbb{R}^n \, ,$$

where $X_r = \{x \in X : |x| \le r\}$.

Prove it.

[1] Additional literature (for especially interested):
M. Pascu (2006) “On the definition of multidimensional generalized Riemann integral”, Bul. Univ. Petrol **LVIII**:2, 9–16.
(*Research level*) D. Maharam (1988) “Jordan fields and improper integrals”, J. Math. Anal. Appl. **133**, 163–194.
Z. Kánnai (2008) “Uniform convergence for convexification of dominated pointwise convergent continuous functions”, arXiv:0809.0393.

9b4 Definition. A set $A \subset \mathbb{R}^n$ is *locally Jordan measurable* if $A \cap E$ is Jordan measurable for all Jordan measurable $E \subset \mathbb{R}^n$.

9b5 Exercise. A set $A \subset \mathbb{R}^n$ is locally Jordan measurable if and only if A_r is Jordan measurable for all r.

Prove it.

9b6 Lemma. Locally Jordan measurable sets are an algebra of sets (in $\mathbb{R}^n$). That is, $\emptyset$, $\mathbb{R}^n \setminus A$, $A \cap B$ (and therefore also $\mathbb{R}^n$, $A \cup B$ and $A \setminus B$) are locally Jordan measurable whenever A, B are.

Proof. For every Jordan measurable E the sets $\emptyset \cap E = \emptyset$, $(\mathbb{R}^n \setminus A) \cap E = E \setminus (A \cap E)$ and $(A \cap B) \cap E = (A \cap E) \cap (B \cap E)$ are Jordan measurable by 6j4. $\square$

Generally, v_* is not additive (even for bounded sets) but superadditive:

$$v_*(A \uplus B) \ge v_*(A) + v_*(B) \tag{9b7}$$

for all $A, B \in \mathbb{R}^n$, $A \cap B = \emptyset$ (since $v_*(A \uplus B) \ge v(E \uplus F) = v(E) + v(F)$ for all Jordan $E \subset A$, $F \subset B$).

9b8 Lemma. The restriction of v_* to the algebra of locally Jordan sets is additive.

Proof. Let A, B be locally Jordan sets, and $A \cap B = \emptyset$. We have

$$v_*(A) \overset{9b3}{=} \lim_{r\to\infty} v_*(A_r) = \lim_{r\to\infty} v(A_r)\,.$$

The same holds for B and $A \uplus B$. It remains to take the limit in $v(A_r \uplus B_r) = v(A_r) + v(B_r)$. $\square$

For a locally Jordan A, $v_*(A)$ may be called the volume of A.

9b9 Definition. A *locally volume zero* set is a locally Jordan measurable set $Z \subset \mathbb{R}^n$ such that $v_*(Z) = 0$.

By 9b3 and 9b5,

$$Z \text{ is locally volume zero} \quad \Longleftrightarrow \quad \forall r \ \big(Z_r \text{ is volume zero}\big)\,. \tag{9b10}$$

Here is a generalization of 6k4.

9b11 Lemma. A set $A \subset \mathbb{R}^n$ is locally Jordan measurable if and only if its boundary is locally volume zero.

Proof. By 9b5, (9b10) and 6k4 it is sufficient to prove that

$$\forall r \ \big(\partial(A_r) \text{ is volume zero}\big) \quad \Longleftrightarrow \quad \forall r \ \big((\partial A)_r \text{ is volume zero}\big) \,.$$

On one hand, $(\partial A)_r \subset \partial(A_s)$ for $r < s$. On the other hand, $\partial(A_r) \subset (\partial A)_r \cup \{x : |x| = r\}$. (Alternatively: $(\partial A_r)\Delta(\partial A)_r \subset \{x : |x| = r\}$.) $\square$

Similarly to 9b4, for arbitrary $X \subset \mathbb{R}^n$, a set $A \subset X$ is called *locally Jordan measurable in* X if $A \cap E$ is Jordan measurable for all Jordan measurable $E \subset X$. (Note that X is locally Jordan in X, even if not in $\mathbb{R}^n$.) If A is locally Jordan in $\mathbb{R}^n$ then $A \cap X$ is locally Jordan in X.[1] More generally, if $X \subset Y \subset \mathbb{R}^n$ and $A \subset Y$ is locally Jordan in Y then $A \cap X$ is locally Jordan in X.

Similarly to 9b6, sets locally Jordan in X are an algebra of sets (in X). That is, $\emptyset$, $X \setminus A$, $A \cap B$ (and therefore also X, $A \cup B$ and $A \setminus B$) are locally Jordan measurable in X whenever A, B are. (Prove it.)

Similarly to 9b8, the restriction of v_* to this algebra of sets is additive (and may be called the volume in X). However, the proof is different, since A_r are now irrelevant.

9b12 Lemma. The restriction of v_* to the algebra of sets locally Jordan in X is additive.

Proof. Let A, B be locally Jordan in X, and $A \cap B = \emptyset$. By (9b7) it is sufficient to prove that $v_*(A \uplus B) \le v_*(A) + v_*(B)$. Let $E \subset A \uplus B$ be Jordan, then $v(E) = v\big((E \cap A) \uplus (E \cap B)\big) = v(E \cap A) + v(E \cap B) \le v_*(A) + v_*(B)$. $\square$

Similarly to 9b9, a set of *locally volume zero in* X is a locally Jordan in X set $Z \subset X$ such that $v_*(Z) = 0$.

A counterpart of 9b11 holds but also needs a different proof.

9b13 Lemma. A set $A \subset X$ is locally Jordan in X if and only if $\partial A \cap X$ is locally volume zero in X.

Proof. By 9b11 it is sufficient to prove that

$$v^*\big(\partial(A \cap E)\big) = 0 \quad \Longleftrightarrow \quad v^*(\partial A \cap E) = 0$$

for every Jordan $E \subset X$. To this end it is sufficient to check that

$$\big(\partial(A \cap E)\big)\Delta(\partial A \cap E) \subset \partial E \qquad \text{for all } A, E \subset \mathbb{R}^n \,.$$

[1] Do such $A \cap X$ exhaust all sets locally Jordan in X? Generally, not (try $X = \mathbb{R} \setminus \mathbb{Q}$). For an open X, I do not know. (I guess, not.)

In other words, that $\partial(A \cap E) \cap U = (\partial A \cap E) \cap U$ both for $U = E^\circ$ and for $U = (\mathbb{R}^n \setminus E)^\circ$. The latter case is trivial: $\emptyset = \emptyset$. Let $U = E^\circ$. We note that "boundary" is a local notion: $A \cap U = B \cap U$ implies $\partial A \cap U = \partial B \cap U$ (given that U is open). We have $(A \cap E) \cap U = A \cap U$, thus $\partial(A \cap E) \cap U = \partial A \cap U = (\partial A \cap E) \cap U$. □

In particular, for $A = X$ we have

$$\partial X \cap X \text{ is locally volume zero in } X \tag{9b14}$$

for all $X \subset \mathbb{R}^n$. (Even if $\partial X = \mathbb{R}^n$.) Throwing away this set we get $X \setminus (X \cap \partial X) = X^\circ$. It means that, without loss of generality, we may restrict ourselves to open sets $G \subset \mathbb{R}^n$ (rather than arbitrary sets $X \subset \mathbb{R}^n$), sets locally Jordan in G, and their volumes in G. Similarly to 6k1,

$$v_*(X^\circ) = v_*(X)\,. \tag{9b15}$$

And do not forget that an open set need not be Jordan (even if bounded and diffeomorphic to a disk, as noted in Sect. 8a), nor locally Jordan.

9c Monotone convergence of volumes

Given sets $X, X_1, X_2, \dots$ we write $X_i \uparrow X$ when $X_1 \subset X_2 \subset \dots$ and $\cup_i X_i = X$. Similarly, we write $X_i \downarrow X$ when $X_1 \supset X_2 \supset \dots$ and $\cap_i X_i = X$.

9c1 Theorem. *(Monotone convergence theorem for volumes)* Let $X \subset \mathbb{R}^n$, sets $A_i \subset X$ be locally Jordan in X, and $A_i \uparrow X$, then

$$v_*(A_i) \uparrow v_*(X) \qquad \text{as } i \to \infty\,.$$

9c2 Remark. By 6k11, for every Jordan set E,

$$v(E) = \sup_{K \subset E} v(K)$$

where K runs over compact Jordan sets (moreover, closed pixelated sets suffice). Thus, (9b2) is equivalent to

$$v_*(A) = \sup\{v(K) : \text{compact Jordan } K \subset A\} \in [0, \infty]\,.$$

9c3 Lemma. If $X_i \subset \mathbb{R}^n$, $X_i \downarrow \emptyset$ and $v_*(X_1) < \infty$, then $v_*(X_i) \downarrow 0$ as $i \to \infty$.

Proof. Assume the contrary: $v_*(X_i) \downarrow 2\varepsilon$ for some $\varepsilon > 0$. For each i there exists a compact Jordan set $K_i \subset X_i$ such that $v(K_i) \ge v_*(X_i) - 2^{-i}\varepsilon$. By compactness there exists m such that $K_1 \cap \dots \cap K_m = \emptyset$. We have $K_m = (K_m \setminus K_1) \cup \dots \cup (K_m \setminus K_{m-1})$, thus $v(K_m) \le v(K_m \setminus K_1) + \dots + v(K_m \setminus K_{m-1})$.

For each $i = 1, \dots, m-1$ we have

$$\begin{gathered} K_i \uplus (K_m \setminus K_i) = K_i \cup K_m \subset X_i \cup X_m = X_i\,; \\ v(K_i) + v(K_m \setminus K_i) \le v_*(X_i)\,; \\ v(K_m \setminus K_i) \le v_*(X_i) - v(K_i) \le 2^{-i}\varepsilon\,; \\ v(K_m) \le \sum_{i=1}^{m-1} 2^{-i}\varepsilon < \varepsilon\,. \end{gathered}$$

On the other hand, $v(K_m) \ge v_*(X_m) - 2^{-m}\varepsilon > 2\varepsilon - \varepsilon = \varepsilon$; a contradiction. □

9c4 Lemma. Let $E \subset \mathbb{R}^n$ be a Jordan set, $f : E \to \mathbb{R}$ a bounded function, and $h : E \to \mathbb{R}$ an integrable function. Then

$$\int_{*E} (f+h) = \int_{*E} f + \int_E h\,, \qquad \int^*_E (f+h) = \int^*_E f + \int_E h\,.$$

Proof. On one hand, $^*\!\!\int (f+h) \le {}^*\!\!\int f + {}^*\!\!\int h = {}^*\!\!\int f + \int h$. On the other hand, $^*\!\!\int f = {}^*\!\!\int \big((f+h) + (-h) \big) \le {}^*\!\!\int (f+h) + \int (-h)$, that is, $^*\!\!\int (f+h) \ge {}^*\!\!\int f + \int h$, which proves the second relation. For the first, change the sign. □

9c5 Lemma. Let $E \subset \mathbb{R}^n$ be a Jordan set, $f, g : E \to \mathbb{R}$ bounded functions such that $f + g$ is integrable. Then

$$\int_{*E} f + \int^*_E g = \int_E (f+g) = \int^*_E f + \int_{*E} g\,.$$

Proof. $^*\!\!\int f = {}^*\!\!\int \big((-g) + (f+g) \big) = {}^*\!\!\int (-g) + \int (f+g) = \int (f+g) - {}_*\!\!\int g$. □

9c6 Corollary. Let $E \subset \mathbb{R}^n$ be a Jordan set, then

$$v^*(X) + v_*(E \setminus X) = v(E)$$

for all subsets $X \subset E$.

Proof. $1\!\!1_X + 1\!\!1_{E \setminus X} = 1\!\!1_E$; apply 9c5. □

9c7 Lemma. If $X_i \uparrow X$ then $v_*(X) \le \lim_i v^*(X_i)$ for bounded $X_i \subset \mathbb{R}^n$.

Proof. By (9b2) it is sufficient to prove that $v(E) \le \lim_i v^*(X_i)$ for all Jordan $E \subset X$. We have $X \setminus X_i \downarrow \emptyset$, thus $E \setminus X_i \downarrow \emptyset$. By 9c3, $v_*(E \setminus X_i) \downarrow 0$. By 9c6, $v(E) = v^*(E \cap X_i) + v_*(E \setminus X_i) \le v^*(X_i) + v_*(E \setminus X_i) \to \lim_i v^*(X_i)$. □

Proof of Theorem 9c1. Denote $V = \lim_i v_*(A_i)$. Clearly, $v_*(X) \ge V$; we have to prove that $v_*(X) \le V$, that is, $v(E) \le V$ for all Jordan $E \subset X$.

Lemma 9c7 applied to Jordan sets $E \cap A_i \uparrow E$ gives $v(E) \le \lim_i v(E \cap A_i)$; and $v(E \cap A_i) \le v_*(A_i) \le V$. □

9c8 Corollary. For all Jordan sets $E_1, E_2, \dots \in \mathbb{R}^n$,

$$v_*(E_1 \cup E_2 \cup \dots) \le v(E_1) + v(E_2) + \dots$$

Proof. $v_*(E_1 \cup E_2 \cup \dots) = \lim_i v(E_1 \cup \dots \cup E_i) \le v(E_1) + v(E_2) + \dots$ □

9c9 Corollary. If $a_i \in \mathbb{R}$, $\varepsilon_i > 0$ satisfy $\sum_i \varepsilon_i < 1$ then there exists $t \in (0,1)$ such that $\forall i \;\; t \notin [a_i, a_i + \varepsilon_i]$. Moreover, there exist uncountably many such t.

9c10 Example. *(A simple fact about Diophantine approximation)* Uncountably many real numbers x do not admit rational approximations $x \approx \dfrac{p}{q}$ satisfying $\left| x - \dfrac{p}{q} \right| < \dfrac{1}{4q^3}$.

Indeed, for a given q the set

$$A_q = \left\{ x \in (0,1) : \exists p \; \left| x - \frac{p}{q} \right| < \frac{1}{4q^3} \right\}$$

consists of intervals of total length $\frac{1}{2q^2}$ (namely, $q-1$ intervals of length $\frac{1}{2q^3}$ and two intervals of length $\frac{1}{4q^3}$). Thus, $\sum_q v_1(A_q) = \sum_q \frac{1}{2q^2} = \frac{1}{2} \cdot \frac{\pi^2}{6} < 1$.

Do not think that $v_*(A_1 \cup A_2 \cup \dots) = \lim_i v_*(A_i)$ for *arbitrary* $A_1 \subset A_2 \subset \dots$

9c11 Example. It can happen that $X_i \uparrow \mathbb{R}$ and $v_*(X_i) = 0$ for all i.

Define X_i as consisting of all rational numbers with denominators at most i and all irrational numbers. Then X_i has no interior points, thus $v_*(X_i) = 0$. However, $X_i \uparrow \mathbb{R}$.

9d Improper integral

9d1 Lemma. A bounded function $f : \mathbb{R}^n \to [0, \infty)$ with bounded support is integrable if and only if the set $E = \{(x,t) : 0 < t < f(x)\}$ is Jordan measurable. In this case $\int_{\mathbb{R}^n} f = v(E)$.

Proof. If f is integrable then the set is Jordan, and the equality holds, according to 6h1, 6h2 (and 6j4).

If E is Jordan then f is integrable by Th. 7d1, since $f(x) = \int_{\mathbb{R}} 1\!\!1_E(x,t) \, \mathrm{d}t$. □

We generalize integrability and integral as follows.

9d2 Definition. (a) A function $f : \mathbb{R}^n \to \mathbb{R}$ is *Jordan measurable* if the set

$$\{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}, t < f(x)\}$$

is locally Jordan measurable in $\mathbb{R}^{n+1}$.

(b) A Jordan measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is *integrable* if

$$v_*\big(\{(x,t) : 0 < t < f(x)\}\big) < \infty \quad \text{and} \quad v_*\big(\{(x,t) : f(x) < t < 0\}\big) < \infty\,.$$

In this case its *improper integral* is

$$\int_{\mathbb{R}^n} f = v_*\big(\{(x,t) : 0 < t < f(x)\}\big) - v_*\big(\{(x,t) : f(x) < t < 0\}\big)\,.$$

9d3 Remark. If f is Jordan measurable then the boundary of the set $\{(x,t) : t < f(x)\}$ is locally volume zero by 9b11, thus the graph $\{(x,t) : t = f(x)\}$ is locally volume zero (being a part of the boundary); by 9b6, sets $\{(x,t) : t \le f(x)\}$, $\{(x,t) : t > f(x)\}$, $\{(x,t) : t \ge f(x)\}$ are locally Jordan; also sets $\{(x,t) : 0 < t < f(x)\}$ and $\{(x,t) : f(x) < t < 0\}$ are locally Jordan, since $\mathbb{R}^n \times (0,\infty)$ and $\mathbb{R}^n \times (-\infty,0)$ are.

9d4 Remark. It may happen that $v_*\big(\{(x,t) : 0 < t < f(x)\}\big) = \infty$ and $v_*\big(\{(x,t) : f(x) < t < 0\}\big) < \infty$. Then f is not integrable, and one says that its improper integral is $+\infty$ ($= +\infty -$ real). Similarly, real $- \infty = -\infty$. However, $\infty - \infty$ is undefined.

9d5 Remark. In other words,

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \max(f,0) - \int_{\mathbb{R}^n} \max(-f,0)\,.$$

9d6 Remark. If a Jordan measurable function is bounded, with bounded support, then 9d2(b) is satisfied, since the sets $\{(x,t) : 0 < t < f(x)\}$ and $\{(x,t) : f(x) < t < 0\}$ are bounded. In this case the improper integral is equal to the proper integral by 9d1 and 9d5.

Similarly to 6j5, given an integrable $f : \mathbb{R}^n \to \mathbb{R}$ and a locally Jordan $A \subset \mathbb{R}^n$, we define

$$\int_A f = \int_{\mathbb{R}^n} f \cdot 1\!\!1_A = \\ = v_*\big(\{(x,t) : x \in A, 0 < t < f(x)\}\big) - v_*\big(\{(x,t) : x \in A, f(x) < t < 0\}\big)\,, \tag{9d7}$$

taking into account that $A \times \mathbb{R}$ is locally Jordan in $\mathbb{R}^{n+1}$ (recall 7d4).

Similarly to (6j6), using 9b8 we see that the improper integral is an additive set function,

(9d8) $$\int_{A \uplus B} f = \int_A f + \int_B f \, .$$

By Theorem 9c1,

(9d9) $$\int_A f = \lim_i \int_{A_i} f$$

whenever A and A_i are locally Jordan sets such that $A_i \uparrow A$ (since in this case $A_i \times \mathbb{R} \uparrow A \times \mathbb{R}$).

In practice one often chooses bounded sets A_i such that f is bounded on each A_i; this way an improper integral becomes the limit of proper integrals. Alternatively,

(9d10) $$\int_A f = \lim_i \int_{A_i} f_i \quad \text{where } f_i(x) = \begin{cases} -i & \text{if } f(x) \le -i, \\ f(x) & \text{if } -i \le f(x) \le i, \\ i & \text{if } i \le f(x) \end{cases}$$

(since $A_i \times [-i, i] \uparrow A \times \mathbb{R}$).

9d11 Proposition. $\int_{\mathbb{R}^n} (f+g) = \int_{\mathbb{R}^n} f + \int_{\mathbb{R}^n} g$ for all Jordan measurable $f, g : \mathbb{R}^n \to [0, \infty)$.

Proof. By (9d9) it is sufficient to prove that $\int_E (f+g) = \int_E f + \int_E g$ for every Jordan $E \subset \mathbb{R}^n$.

On one hand, $f_i + g_i \le (f+g)_{2i}$; using linearity of proper integral, $\int_E f_i + \int_E g_i = \int_E (f_i + g_i) \le \int_E (f+g)_{2i}$, which gives $\int_E f + \int_E g \le \int_E (f+g)$.

On the other hand, $(f+g)_i \le f_i + g_i$, thus $\int_E (f+g)_i \le \int_E f_i + \int_E g_i$, which gives $\int_E (f+g) \le \int_E f + \int_E g$. □

9d12 Theorem. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable then $f + g$ is integrable and

$$\int_{\mathbb{R}^n} (f+g) = \int_{\mathbb{R}^n} f + \int_{\mathbb{R}^n} g \, .$$

Proof. First, $\max(f+g, 0) \le \max(f, 0) + \max(g, 0)$; by 9d11, $\int \max(f + g, 0) \le \int \max(f, 0) + \int \max(g, 0) < \infty$. Similarly, $\int \max(-f-g, 0) < \infty$. Thus, $f + g$ is integrable.

Second, $f = \max(f, 0) - \max(-f, 0)$ and $g = \max(g, 0) - \max(-g, 0)$, thus $f + g = \big(\max(f, 0) + \max(g, 0)\big) - \big(\max(-f, 0) + \max(-g, 0)\big)$, but also $f+g = \max(f+g,0) - \max(-f-g,0)$, therefore $\max(f+g,0) + \max(-f,0) + \max(-g,0) = \max(f,0) + \max(g,0) + \max(-f-g,0)$. By 9d11, $\int \max(f+g,0) + \int \max(-f,0) + \int \max(-g,0) = \int \max(f,0) + \int \max(g,0) + \int \max(-f-g,0)$. Using 9d5, $\int (f+g) = \int \max(f+g,0) - \int \max(-f-g,0) = \int \max(f,0) - \int \max(-f,0) + \int \max(g,0) - \int \max(-g,0) = \int f + \int g$. □

Similarly to 9d2, for arbitrary $X \subset \mathbb{R}^n$, a function $f : X \to \mathbb{R}$ is called *Jordan measurable on* X if the set $\{(x,t) : x \in X, t \in \mathbb{R}, t < f(x)\}$ is locally Jordan measurable in $X \times \mathbb{R}$; and then the integral is defined by

(9d13)
$$\int_X f = v_*\big(\{(x,t) : x \in X, 0 < t < f(x)\}\big) - v_*\big(\{(x,t) : x \in X, f(x) < t < 0\}\big)$$

(be it a number, $+\infty$, $-\infty$ or $\infty - \infty$). Similarly to 9d7, 9d8, a function f integrable on X leads to an additive set function on the algebra of sets locally Jordan in X. And again, (9b14) shows that only the interior of X is relevant. Theorem 9d12 and Prop. 9d11 generalize readily to functions $X \to \mathbb{R}$.

If X is locally Jordan in $\mathbb{R}^n$ then $\int_X f$ defined by (9d13) is the same as $\int_X f$ defined by (9d7), that is, $\int_{\mathbb{R}^n} f \cdot 1\!\!1_X$. But be warned: if X is not locally Jordan in $\mathbb{R}^n$ then $\int_{\mathbb{R}^n} f \cdot 1\!\!1_X$ is not defined even for $f = 1\!\!1$; note also that the set function $X \mapsto \int_X f$ is generally not additive; in particular, $\int_X 1\!\!1 = v_*(X)$.

9e Examples: Poisson formula; inequalities

9e1 Example (Poisson)**.** Consider the integral

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y\,.$$

On one hand we may exhaust the plane $\mathbb{R}^2$ by the discs $A_k = \{(x,y) : x^2+y^2 < k^2\}$. In this case,

$$\iint_{A_k} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^k e^{-r^2} r\,\mathrm{d}r = \pi(1-e^{-k^2}) \to \pi\,.$$

On the other hand, consider the exhaustion by the squares $B_k = \{(x,y) : \max(|x|,|y|) < k\}$. We get[1]

$$\iint_{B_k} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \Big(\int_{-k}^k \mathrm{e}^{-x^2}\,\mathrm{d}x\Big)\Big(\int_{-k}^k \mathrm{e}^{-y^2}\,\mathrm{d}y\Big) \to \Big(\int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x\Big)^2\,.$$

[1]Recall 7b5, 7d3.

Juxtaposing the answers, we obtain the celebrated Poisson formula:

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}}$$

The corresponding n-dimensional integral:

$$\boxed{\int_{\mathbb{R}^n} \mathrm{e}^{-\langle Ax,x\rangle}\,\mathrm{d}x = \frac{\pi^{n/2}}{\sqrt{\det A}}} \tag{9e2}$$

for every positive symmetric $n \times n$ matrix A.

First, observe that

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2}\,\mathrm{d}x = \biggl(\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t \biggr)^n = \pi^{n/2}\,. \tag{9e3}$$

Also observe that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-at^2}\,\mathrm{d}t = \sqrt{\frac{\pi}{a}}\,. \tag{9e4}$$

If the matrix of A is diagonal, then (9e2) follows from (9e4). In general we diagonalize it by choosing an orthonormal basis appropriately.

9e5 Example. Given $\alpha > 0$, we consider $f(x) = |x|^{-\alpha}$ for $x \in \mathbb{R}^n \setminus \{0\}$.

First, let $U = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$. We split the punctured ball into the layers $C_k = \{x : 2^{-k} < |x| \le 2^{1-k}\}$, $k \ge 1$. If $x \in C_k$ then the integrand is between $2^{\alpha(k-1)}$ and $2^{\alpha k}$. Also, $v(C_k) = 2^{-nk} v(C_1)$. Hence the integral $\int_{0<|x|<1} \frac{\mathrm{d}x}{|x|^\alpha}$ converges or diverges simultaneously with the series $\sum_{k\ge1} 2^{(\alpha-n)k}$. We see that the integral converges if $\alpha < n$ and diverges otherwise.

Second, let $U = \{x \in \mathbb{R}^n : |x| > 1\}$. We use a similar decomposition into the layers $\{2^k \le |x| < 2^{k+1}\}$ and obtain the series $\sum_{k\ge1} 2^{(n-\alpha)k}$. Hence, the second integral converges iff $\alpha > n$.

Thus, $\int_{\mathbb{R}^n\setminus\{0\}} f = \infty$ for all $\alpha \in (0, \infty)$.

9e6 Example. Given $\alpha > 0$, we consider the function $f : (x, y) \mapsto (1 - x^2 - y^2)^{-\alpha}$ on the disk $U = \{(x, y) : x^2 + y^2 < 1\}$. We take some $\varepsilon_k \downarrow 0$ and exhaust U by $G_k = \{(x, y) : x^2 + y^2 < (1 - \varepsilon_k)^2\}$. We have

$$\int_{G_k} f = \iint_{x^2+y^2<(1-\varepsilon_k)^2} \frac{\mathrm{d}x\mathrm{d}y}{(1 - x^2 - y^2)^\alpha} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{1-\varepsilon_k} \frac{r\mathrm{d}r}{(1 - r^2)^\alpha} =$$

$$= 2\pi \cdot \frac{1}{2} \int_0^{1-\varepsilon_k} \frac{\mathrm{d}s}{(1 - s)^\alpha} = \pi \int_{\varepsilon_k}^1 \frac{dt}{t^\alpha} \to \frac{\pi}{1 - \alpha}$$

if $\alpha < 1$, otherwise ∞. Thus,

$$\iint_{x^2+y^2<1} \frac{\mathrm{d}x\mathrm{d}y}{(1-x^2-y^2)^\alpha} = \begin{cases} \frac{\pi}{1-\alpha} & \text{for } \alpha \in (0,1), \\ \infty & \text{for } \alpha \in [1,\infty). \end{cases}$$

9e7 Exercise. Compute the integral $\int_Q \frac{dx}{|x|}$ where $Q = (0,1)^2$ is the unit square in $\mathbb{R}^2$.
Hint: $\int \frac{\mathrm{d}\varphi}{\cos\varphi} = \int \frac{\mathrm{d}\sin\varphi}{1-\sin^2\varphi}$.

9e8 Exercise. Compute $\iint_{\mathbb{R}^2} |ax+by| \mathrm{e}^{-(x^2+y^2)/2} \, \mathrm{d}x\mathrm{d}y$.
Hint: choose a convenient orthonormal basis.

9e9 Exercise. Compute $\int_{\mathbb{R}^n} |\langle x,a\rangle|^p \mathrm{e}^{-|x|^2} \, \mathrm{d}x$ for $a \in \mathbb{R}^n$ and $p \in (-1,\infty)$.
Hint: choose a convenient orthonormal basis.

9e10 Exercise. Prove that $\int_{\mathbb{R}^3} \frac{\mathrm{d}\xi}{|x-\xi|^2|y-\xi|^2} = \frac{c}{|x-y|}$ for some constant $c \in (0,\infty)$.
Hint: a linear change of variables.

9e11 Exercise. For which values of p and q does the integral $\iint_{|x|+|y|>1} \frac{dxdy}{|x|^p+|y|^q}$ converge?

9e12 Exercise. Find the sign of the integral $\iint_{\max(|x|,|y|)<1} \ln(x^2+y^2) \, \mathrm{d}x\mathrm{d}y$.
Hint: $\int_0^{1/\cos\varphi} r \ln r \, \mathrm{d}r < 0$ for $\varphi \in [0,\pi/4]$.

9e13 Exercise. Whether the integrals $\iint_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}y}{1+x^{10}y^{10}}$ and $\iint_{\mathbb{R}^2} \mathrm{e}^{-(x+y)^4} \, \mathrm{d}x\mathrm{d}y$ converge or diverge?

Some inequalities

Here are the integral versions of the classical inequalities of Cauchy-Schwarz,[1] [2] Hölder[3] [4] and Minkowski.[5] [6]

By $\tilde{L}^p(U)$ we denote[7] (for a given open set $U \subset \mathbb{R}^n$ and a number $p \in [1,\infty)$) the set of all functions f Jordan measurable on U, satisfying $\int_U |f|^p < \infty$. For such f we define

$$\|f\|_p = \left(\int_U |f|^p \right)^{1/p}.$$

[1] $|a_1b_1 + \dots + a_nb_n| \le \sqrt{|a_1|^2 + \dots + |a_n|^2}\sqrt{|b_1|^2 + \dots + |b_n|^2}$, that is, $|\langle a,b\rangle| \le |a||b|$.
[2] See also 6d16 (and 6d17(b)).
[3] $|a_1b_1 + \dots + a_nb_n| \le (|a_1|^p + \dots + |a_n|^p)^{1/p}(|b_1|^q + \dots + |b_n|^q)^{1/q}$.
[4] See also (3i2).
[5] $(|a_1+b_1|^p + \dots + |a_n+b_n|^p)^{1/p} \le (|a_1|^p + \dots + |a_n|^p)^{1/p} + (|b_1|^p + \dots + |b_n|^p)^{1/p}$.
[6] See also 1e15.
[7] The widely used notation L^p is reserved for the corresponding notion in the framework of Lebesgue integration.

9e14 Claim (Cauchy-Schwarz)**.** Suppose $f, g \in \tilde{L}^2(U)$. Then $fg \in \tilde{L}_1(U)$ and $\big| \int_U fg \big| \le \|f\|_2 \|g\|_2$.

9e15 Claim (Hölder)**.** More generally, $fg \in \tilde{L}_1(U)$ and $\big| \int_U fg \big| \le \|f\|_p \|g\|_q$ whenever $f \in \tilde{L}^p(U)$, $g \in \tilde{L}^q(U)$, $\frac{1}{p} + \frac{1}{q} = 1$.

9e16 Claim (Minkowski)**.** If $f, g \in \tilde{L}^p(U)$ then $f+g \in \tilde{L}^p(U)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

9e17 Exercise. Prove 9e15.
Hint: use the inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \in [0, \infty)$.

9e18 Exercise. Prove 9e16.
Hint: start with $|a+b|^p \le |a||a+b|^{p-1} + |b||a+b|^{p-1}$, then use Hölder's inequality.
Or, alternatively: if $\|f\|_p \le 1$ and $\|g\|_p \le 1$ then $\|cf + (1-c)g\|_p \le 1$ for all $c \in [0, 1]$, since the function $t \mapsto |t|^p$ is convex.
Still another approach: $\|f\|_p = \sup\{\int fg : \|g\|_q \le 1\}$.

9f Change of variables in improper integral

First we generalize Prop. 8a2.

9f1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $A \subset U$. Then A is locally Jordan in U if and only if $\varphi(A)$ is locally Jordan in V.

9f2 Lemma. Let $E_1, E_2, \dots$ be Jordan sets, and a bounded $X \subset \mathbb{R}^n$ satisfy $v^*(X \Delta E_i) \to 0$ as $i \to \infty$. Then X is a Jordan set, and $v(E_i) \to v(X)$.

Proof. Apply 6d15 to $\mathbb{1}_X$ and $\mathbb{1}_{E_i}$. □

9f3 Lemma. Let $X \subset \mathbb{R}^n$, sets $A_i \subset X$ be locally Jordan in X, and $A_i \uparrow X$. Then a set $B \subset X$ is locally Jordan in X if and only if $B \cap A_i$ is locally Jordan in A_i for each i, and in this case $v_*(B \cap A_i) \uparrow v_*(B)$.

Proof. Let $B \cap A_i$ be locally Jordan in A_i for each i; we have to prove that B is locally Jordan in X (the converse implication being trivial), that is, $B \cap E$ is Jordan for every Jordan $E \subset X$.

Sets $F_i = E \setminus A_i$ are Jordan measurable, and $F_i \downarrow \emptyset$, thus $v(F_i) \to 0$ by 9c3. Sets $B \cap A_i \cap E$ are Jordan measurable, and $(B \cap E) \setminus (B \cap A_i \cap E) \subset F_i$, therefore $v^*\big((B \cap E) \setminus (B \cap A_i \cap E)\big) \to 0$. By 9f2, $B \cap E$ is Jordan measurable. Thus, B is locally Jordan in X. By Th. 9c1, $v_*(B \cap A_i) \uparrow v_*(B)$. □

9f4 Corollary. Let $G \subset \mathbb{R}^n$ be open and $A \subset G$. Then A is locally Jordan in G if and only if $A \cap E$ is Jordan for every Jordan E contained in a compact subset of G, and the supremum over these E of $v(A \cap E)$ is equal to $v_*(A)$.

Proof of Prop. 9f1. Let A be locally Jordan in U, and $B = f(A) \subset V$; we'll prove that B is locally Jordan in V (the converse being the same for φ^{-1}). Let $F \subset V$ be a Jordan set contained in a compact subset of V; by 9f4 it is sufficient to prove that $B \cap F$ is Jordan. By 8a2, the set $E = \varphi^{-1}(F)$ is Jordan measurable and contained in a compact subset of U. Thus, $A \cap E$ is Jordan (and still contained in a compact subset of U). By 8a2 (again), $f(A \cap E)$ is Jordan. It remains to note that $f(A \cap E) = B \cap F$. $\square$

Now we generalize Theorem 8a5 and Corollary 8a6.

9f5 Theorem. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then f is Jordan measurable on V if and only if $f \circ \varphi$ is Jordan measurable on U, and in this case

$$\int_V f = \int_U (f \circ \varphi) |\det D\varphi| \, .$$

9f6 Remark. The equality may be "real = real", "$+\infty = +\infty$", "$-\infty = -\infty$", or "$\infty - \infty = \infty - \infty$".

Proof. The mapping $(\varphi \times \mathrm{id}) : U \times \mathbb{R} \to V \times \mathbb{R}$, $(\varphi \times \mathrm{id})(x, t) = (\varphi(x), t)$, is also a diffeomorphism, since $D(\varphi \times \mathrm{id}) = \left(\begin{array}{c|c} D\varphi & 0 \\ \hline 0 & \mathrm{id} \end{array}\right)$. Prop. 9f1 applied to $\varphi \times \mathrm{id}$ shows that the set $\{(x, t) : x \in U, t < (f \circ \varphi)(x)\}$ is locally Jordan in $U \times \mathbb{R}$ if and only if the set $\{(y, t) : y \in V, t < f(y)\}$ is locally Jordan in $V \times \mathbb{R}$. Thus, $f \circ \varphi$ is Jordan measurable on U if and only if f is Jordan measurable on V.

It remains to prove the equality of the integrals. By 9d5 we may assume that $f \ge 0$. We take compact Jordan sets $E_i \subset U$ such that $E_i \uparrow U$, and denote $F_i = \varphi(E_i)$. By Theorem 8a5,

$$\forall i \quad \int_{F_i} f_i = \int_{E_i} (f_i \circ \varphi) |\det D\varphi| \, ;$$

here $f_i(x) = \min(f(x), i)$. By 9d1,

$$\int_{F_i} f_i = v\big(A \cap (F_i \times [0, i]\big)$$

where $A = \{(x,t) : 0 < t < f(x)\} \subset V \times \mathbb{R}$. By Th. 9c1, $v\big(A \cap (F_i \times [0,i])\big) \uparrow v_*(A)$, that is,

$$\int_{F_i} f_i \uparrow \int_V f \, .$$

Similarly, $\int_{E_i} (f_i \circ \varphi) |\det D\varphi| \uparrow \int_U (f \circ \varphi) |\det D\varphi|$. Thus, $\int_V f = \int_U (f \circ \varphi) |\det D\varphi|$. $\square$

9g Examples: Newton potential

The gravitational force $F(x)$ exerted by the particle of mass μ at point ξ on a particle of mass m at point x is

$$F(x) = -G \frac{m\mu}{|x-\xi|^3}(x - \xi) = Gm\nabla U(x)$$

where the function $U : x \mapsto \frac{\mu}{|x-\xi|}$ is called the Newton (or gravitational) potential and G is the gravitational constant.[1] [2] This is the celebrated Newton law of gravitation. The reason to replace the force F by the potential U is simple: it is easier to work with scalar functions than with the vector ones.[3]

What happens if we have a system of point masses μ_1, ..., μ_N at points ξ_1, ..., ξ_N? The forces are to be added, and the corresponding potential is

$$U(x) = \sum_{j=1}^N \frac{\mu_j}{|x - \xi_j|} \, .$$

Now, suppose that the masses are distributed with continuous density $\mu(\xi)$ over a portion Ω of the space. Then the Newton potential is

$$U(x) = \int_\Omega \frac{\mu(\xi) \, \mathrm{d}\xi}{|\xi - x|}$$

(the integral being three-dimensional), and the corresponding gravitational force (after normalization $G = 1$, $m = 1$) is again $F = \nabla U$.

---

[1] $G = 6.6743 \cdot 10^{-11}\,\mathrm{N(m/kg)^2}$; that is, if $m = \mu = 1\,\mathrm{kg}$ and $|x - \xi| = 1\,\mathrm{m}$ then $|F| = 6.6743 \cdot 10^{-11}$ newtons.

[2] Mathematicians usually omit not only the physical constant G but also the minus sign; in physics, $F = -\nabla U$ and $U(x) = -G\mu \frac{1}{|x-\xi|}$ (for $m = 1$).

[3] Knowing the force F one can write down the differential equations of motion of the particle (Newton's second law) $m\ddot{x} = F$, or $\ddot{x} = G\nabla \frac{\mu}{|x-\xi|}$ (note that m does not matter). Then one hopes to integrate these equations and to find out where is the particle at time t.

Let us compute the Newton potential of the homogeneous mass distribution (that is, $\mu(\xi) = 1$) within the ball B_R of radius R centered at the origin:

$$U(x) = \int_{B_R} \frac{\mathrm{d}\xi}{|x - \xi|} \, .$$

By symmetry U is a radial function, that is, depends only on $|x|$.

9g1 Exercise. Check this!

Thus, it suffices to compute U at the point $x = (0, 0, z)$, $z \ge 0$. Using the spherical coordinates $\xi_1 = r \sin\theta \cos\varphi$, $\xi_2 = r \sin\theta \sin\varphi$, $\xi_3 = r \cos\theta$ we have[1]

$$U = \int_0^R \mathrm{d}r \, 2\pi \int_0^\pi \frac{r^2 \sin\theta \, \mathrm{d}\theta}{\sqrt{(z - r\cos\theta)^2 + r^2 \sin^2\theta}} = \\ = \int_0^R \mathrm{d}r \, \underbrace{2\pi \int_0^\pi \frac{r^2 \sin\theta \, \mathrm{d}\theta}{\sqrt{z^2 - 2zr\cos\theta + r^2}}}_{V} \, .$$

The under-braced expression V is the Newton potential of the homogeneous sphere of radius r. We compute V using the variable

$$t = \sqrt{z^2 - 2zr\cos\theta + r^2} \, .$$

Then $|z - r| < t < z + r$, and $t \, \mathrm{d}t = zr \sin\theta \, \mathrm{d}\theta$. We get

$$V = 2\pi r^2 \int_{|z-r|}^{z+r} \frac{t \, \mathrm{d}t}{zr \cdot t} = \frac{2\pi r}{z} (z + r - |z - r|) = 4\pi \frac{r}{z} \min(r, z) \, .$$

Now we easily find U by integration:

$$U = \int_0^R V \, \mathrm{d}r \, .$$

Outside the ball $z > R$, thus

$$U = 4\pi \int_0^R \frac{r^2}{z} \, \mathrm{d}r = \frac{4\pi R^3}{3z} \, .$$

Inside the ball $z < R$, thus[2]

$$U = 4\pi \biggl(\int_0^z \frac{r^2}{z} \, \mathrm{d}r + \int_z^R r \, \mathrm{d}r \biggr) = 4\pi \Bigl(\frac{z^2}{3} + \frac{R^2}{2} - \frac{z^2}{2} \Bigr) = \frac{2\pi}{3} (3R^2 - z^2) \, .$$

[1] Note that in the case $z < R$ the original integral is improper, and we treat it as iterated! Wait for Sect. 9i for the needed theory.

[2] A wonder: the original improper integral turned into a proper integral.

Finally,

$$U(x) = \begin{cases} \frac{4\pi R^3}{3|x|} & \text{for } |x| \ge R, \\ \frac{2\pi}{3}(3R^2 - |x|^2) & \text{for } |x| \le R. \end{cases}$$

Observe that $4\pi R^3/3$ is exactly the total mass of the ball B_R. That is, together with Newton, we arrived at the conclusion that *the gravitational potential, and hence the gravitational force exerted by the homogeneous ball on a particle is the same as if the whole mass of the ball were concentrated at its center, if the point is outside the ball.* Of course, you heard about this already in the high-school.

Another important conclusion is that the potential V of the homogeneous sphere does not depend on the point x when x is inside the sphere![1] Hence, *the gravitational force is zero inside the sphere.* The same is true for the homogeneous shell $\{\xi : a < |\xi| < b\}$: there is no gravitational force inside the shell.

9g2 Exercise. Check that all the conclusions are true when the mass distribution $\mu(\xi)$ is radial: $\mu(\xi) = \mu(\xi')$ if $|\xi| = |\xi'|$.

9g3 Exercise. Find the potential of the homogeneous solid ellipsoid $(x^2 + y^2)/b^2 + z^2/c^2 \le 1$ at its center.

9g4 Exercise. Find the potential of the homogeneous solid cone of height h and radius of the base r at its vertex.

9g5 Problem. Show that at sufficiently large distances the potential of a solid S is approximated by the potential of a point with the same total mass located at the center of mass of S with an error less than a constant divided by the square of the distance. The potential itself decays as the distance, so the approximation is good: its *relative* error is small.[2]

9h Monotone convergence of integrals

We generalize 9d1 as follows.

9h1 Lemma. Let $f : \mathbb{R}^n \to [0, \infty)$ be a bounded function with bounded support, and $X = \{(x, t) : 0 < t < f(x)\}$. Then ${}_*\int_{\mathbb{R}^n} f = v_*(X)$.

[1] Since V does not depend on z for $z < r$.

[2] This estimate is rather straightforward. A more accurate argument shows that the error is of order constant divided by the *cube* of the distance.

Proof. On one hand, the argument of 6h1 gives $v(E_-) = L(f, P)$ where $E_- = \cup_{C \in P} C \times (0, \inf_C f) \subset X$; the supremum over partitions P gives ${}_*\!\int_{\mathbb{R}^n} f \le v_*(X)$.

On the other hand, for every Jordan set $E \subset X$ we have $\int_{\mathbb{R}} 1\!\!1_E(x, t)\,\mathrm{d}t \le f(x)$ for all x; thus,

$$v(E) = \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}t\, 1\!\!1_E(x, t) \le {}_*\!\!\int_{\mathbb{R}^n} f \,;$$

by 9b1, the supremum over E gives $v_*(X) \le {}_*\!\int_{\mathbb{R}^n} f$. □

The (proper) lower integral ${}_*\!\int_{\mathbb{R}^n} f$ was defined by 6e4 for bounded $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support. Now we define the improper lower integral ${}_*\!\int_X f$ for arbitrary $X \subset \mathbb{R}^n$ and $f : X \to [0, \infty)$ by

$$\text{(9h2)} \qquad {}_*\!\!\int_X f = v_*\big(\{(x, t) : x \in X, 0 < t < f(x)\}\big) \in [0, \infty] \,,$$

note that

$$\text{(9h3)} \qquad {}_*\!\!\int_X f = {}_*\!\!\int_{\mathbb{R}^n} f \cdot 1\!\!1_X$$

and that, by Lemma 9h1, the two definitions do not conflict.

Similarly to (9d9) and (9d10),

$$\text{(9h4)} \qquad {}_*\!\!\int_X f = \lim_i {}_*\!\!\int_{A_i} f$$

whenever sets A_i locally Jordan in X are such that $A_i \uparrow X$; and

$$\text{(9h5)} \qquad {}_*\!\!\int_X f = \lim_i {}_*\!\!\int_{A_i} \min(f(\cdot), i) \,,$$

since $A_i \times [0, i] \uparrow X \times [0, \infty)$, and $A_i \times [0, i]$ are locally Jordan in $X \times \mathbb{R}$. Taking boxes $B_i \uparrow \mathbb{R}^n$ we have, using 9h3, the proper lower integral and (6g7),

$${}_*\!\!\int_X f = {}_*\!\!\int_{\mathbb{R}^n} f \cdot 1\!\!1_X = \lim_i {}_*\!\!\int_{B_i} \min(f, i) \cdot 1\!\!1_X = \sup_i \sup_{h \le \min(f,i) \cdot \mathbf{1}_X} \int_{B_i} h$$

where h runs over all step functions on B_i. It follows that

$$\text{(9h6)} \qquad {}_*\!\!\int_X f = \sup \left\{ \int_{\mathbb{R}^n} h : h \text{ integrable, } h \le f \cdot 1\!\!1_X \right\}.$$

Given functions $f, f_1, f_2, \dots : X \to \mathbb{R}$ we write $f_i \uparrow f$ when $f_1(x) \le f_2(x) \le \dots$ and $f_i(x) \to f(x)$ for all $x \in X$.

Do not think that $f_i \uparrow f$ implies ${}_*\!\int f_i \uparrow {}_*\!\int f$; it does not, even if $f_i : \mathbb{R} \to \{0, 1\}$. For a counterexample recall 9c11.

9h7 Theorem. *(Monotone convergence theorem for integrals)* Let $X \subset \mathbb{R}^n$ be a set, $f_i : X \to [0,\infty)$ functions Jordan measurable on X, $f_i \uparrow f$, $f : X \to [0,\infty)$. Then $\int_X f_i \uparrow {}_*\!\int_X f$.

Proof. Sets $A_i = \{(x,t) : x \in X, 0 < t < f_i(x)\}$ are locally Jordan in $X \times \mathbb{R}$, and $A_i \uparrow A = \{(x,t) : x \in X, 0 < t < f(x)\}$. By Theorem 9c1, $v_*(A_i) \uparrow v_*(A)$. By (9d13), $\int_X f_i = v_*(A_i)$. By (9h2), ${}_*\!\int_X f = v_*(A)$. $\square$

9i Iterated improper integral

9i1 Lemma. For every $f : \mathbb{R}^{m+n} \to [0,\infty)$,

$$ {}_*\!\int_{\mathbb{R}^{n+m}} f \le {}_*\!\int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} f_x \Big) . $$

Proof. By (9h6) it is sufficient to prove that

$$ \int_{\mathbb{R}^{n+m}} h \le {}_*\!\int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} f_x \Big) $$

for all integrable h such that $h \le f$. Using Theorem 7d1,

$$ \int_{\mathbb{R}^{n+m}} h = \int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} h_x \Big) \le {}_*\!\int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} f_x \Big) . $$

$\square$

Do not think that the equality holds for all f. For a counterexample take f of 7c4 and consider $1 - f$.

9i2 Theorem. (Iterated improper integral for positive functions)

Let functions $f_i : \mathbb{R}^{n+m} \to [0,\infty)$ be Jordan measurable, $f_i \uparrow f$, $f : \mathbb{R}^{n+m} \to [0,\infty)$. Then

$$ {}_*\!\int_{\mathbb{R}^{n+m}} f = {}_*\!\int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} f_x \Big) . $$

9i3 Exercise. Let $X \subset \mathbb{R}^n$, $f_i : X \to [0,\infty)$, $f_i \downarrow 0$ (pointwise), and ${}_*\!\int_X f_1 < \infty$; then ${}_*\!\int_X f_i \downarrow 0$.

Prove it.[1]

9i4 Exercise. ${}_*\!\int (f+g) \le {}_*\!\int f + {}^*\!\int g$ for all bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

Prove it.[2]

[1]Hint: use 9c3.

[2]Hint: given integrable $h \le f + g$, apply 9c5 to f and $h - f$.

9i5 Exercise. If $f_i \uparrow f$ then $_*\!\int_{\mathbb{R}^n} f \le \lim_i {}^*\!\!\int_{\mathbb{R}^n} f_i$ for bounded functions $f_i : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

Prove it.[1]

Proof of Theorem 9i2. By 9i1 it is sufficient to prove "$\ge$". We take boxes $B_i \uparrow \mathbb{R}^n$, define $g_i : \mathbb{R}^{n+m} \to [0,\infty)$ by

$$g_i(x) = \min\bigl(f_i(x), i\bigr) \cdot 1\!\!1_{B_i}(x)$$

and note that $g_i \uparrow f$, $_*\!\int g_i \uparrow {}_*\!\int f$ (recall (9h5)). We define $\varphi_i, \varphi, \psi : \mathbb{R}^n \to [0,\infty]$ by

$$\varphi_i(x) = {}^*\!\!\int_{\mathbb{R}^m} (g_i)_x \,, \quad \varphi_i \uparrow \varphi \,, \quad \psi(x) = {}_*\!\int_{\mathbb{R}^m} f_x \,.$$

We have to prove that $_*\!\int_{\mathbb{R}^n} \psi \le {}_*\!\int_{\mathbb{R}^{n+m}} f$.

By 9d1, each g_i is integrable. By Theorem 7d1, each φ_i is integrable, and $\int_{\mathbb{R}^{n+m}} g_i = \int_{\mathbb{R}^n} \varphi_i$. By Theorem 9h7, $\int \varphi_i \uparrow {}_*\!\int \varphi$. Applying 9i5 to $(g_i)_x \uparrow f_x$ we get $\psi \le \varphi$. Thus

$$_*\!\int_{\mathbb{R}^n} \psi \le {}_*\!\int_{\mathbb{R}^n} \varphi = \lim_i \int \varphi_i = \lim_i \int_{\mathbb{R}^{n+m}} g_i = {}_*\!\int_{\mathbb{R}^{n+m}} f \,.$$

□

In practice, the function $x \mapsto {}_*\!\int f_x$ usually is Jordan measurable. But in general this is not the case, even if f is continuously differentiable and $f(x,y) \to 0$, $\nabla f(x,y) \to 0$ as $x^2 + y^2 \to \infty$.

9i6 Example. Similarly to 8e6 we choose disjoint intervals $[s_k, t_k] \subset [0,1]$, whose union is dense on $[0,1]$, such that $\sum_k (t_k - s_k) = a \in (0,1)$, define $f : \mathbb{R}^2 \to [0,\infty)$ by

$$f(x,y) = \sum_{k=1}^{\infty} 1\!\!1_{[s_k,t_k]}(x) 1\!\!1_{[k,k+1]}(y)$$

and observe that

$$\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \sum_{k=1}^{\infty} 1\!\!1_{[s_k,t_k]}(x) = \psi(x) \,, \qquad {}_*\!\int \psi = a < 1 = {}^*\!\!\int \psi \,.$$

In order to get $f(x,y) \to 0$ (as $x^2 + y^2 \to \infty$) we may take

$$f(x,y) = \sum_{k=1}^{\infty} \frac{1}{k} 1\!\!1_{[s_k,t_k]}(x) 1\!\!1_{[k,2k]}(y) \,.$$

[1] Hint: similar to 9c7 (with $\max(h - f_i, 0)$ in place of $E \setminus X_i$).

In order to get $f \in C^1(\mathbb{R}^2)$ we may take

$$f(x,y) = \sum_{k=1}^\infty \frac{1}{k} g\Big(\frac{x-s_k}{t_k-s_k} \Big) h\Big(\frac{y-k}{k} \Big)$$

with appropriate $g,h \in C^1(\mathbb{R})$. In order to get also $Df \to 0$ we may take

$$f(x,y) = \sum_{k=1}^\infty \frac{1}{k} (t_k-s_k) g\Big(\frac{x-s_k}{t_k-s_k} \Big) h\Big(\frac{(t_k-s_k)(y-k)}{k} \Big) \, .$$

9i7 Corollary. If $f : \mathbb{R}^{n+m} \to [0,\infty)$ is Jordan measurable then

$$\int_{*\,\mathbb{R}^n} \mathrm{d}x \int_{*\,\mathbb{R}^m} \mathrm{d}y \, f(x,y) = \int_{\mathbb{R}^{n+m}} f = \int_{*\,\mathbb{R}^m} \mathrm{d}y \int_{*\,\mathbb{R}^n} \mathrm{d}x \, f(x,y) \, .$$

Proof. Apply Theorem 9i2 to $f_i = f$ (and then consider also $\tilde f(y,x) = f(x,y)$). $\square$

9i8 Corollary. For every open set $G \subset \mathbb{R}^{n+m}$,

$$v_*(G) = \int_{*\,\mathbb{R}^n} v_*(G_x) \, \mathrm{d}x$$

where $G_x = \{ y : (x,y) \in G \} \subset \mathbb{R}^m$.

Proof. We have $E_i \uparrow G$ for some Jordan (moreover, pixelated) sets E_i; thus $1\!\!1_{E_i} \uparrow 1\!\!1_G$, and Theorem 9i2 applies. $\square$

This way we can calculate the volume of an open set G even if G is not Jordan measurable, and even if the function $x \mapsto v_*(G_x)$ is not Jordan measurable (which can happen, as shown by 9i6).

9i9 Corollary. For every compact set $K \subset \mathbb{R}^{n+m}$,

$$v^*(K) = {\int^*_{\mathbb{R}^n}} v^*(K_x) \, \mathrm{d}x$$

where $K_x = \{ y : (x,y) \in K \} \subset \mathbb{R}^m$.

Proof. We take boxes $B_1 \subset \mathbb{R}^n$, $B_2 \subset \mathbb{R}^m$, $B = B_1 \times B_2 \subset \mathbb{R}^{n+m}$ such that $K \subset B^\circ$. By 9c6, $v^*(K) + v_*(B^\circ \setminus K) = v(B^\circ)$. We apply 9i8 to the open set $G = B^\circ \setminus K$, note that $G_x = B_2^\circ \setminus K_x$ for $x \in B_1^\circ$ (but $\emptyset$ otherwise) and get

$$v_*(B^\circ \setminus K) = \int_{*\,B_1} v_*(B_2^\circ \setminus K_x) \, \mathrm{d}x \, ,$$

that is,

$$v(B^\circ) - v^*(K) = {}_*\!\!\int_{B_1} \big(v(B_2^\circ) - v^*(K_x) \big) \, \mathrm{d}x \, .$$

By 9c5,

$${}_*\!\!\int_{B_1} \big(v(B_2^\circ) - v^*(K_x) \big) \, \mathrm{d}x + {}^*\!\!\!\int_{B_1} v^*(K_x) \, \mathrm{d}x = \int_{B_1} v(B_2^\circ) \, \mathrm{d}x = v(B^\circ) \, .$$

Thus, $v(B^\circ) - v^*(K) = v(B^\circ) - {}^*\!\!\!\int_{B_1} v^*(K_x)\big) \, \mathrm{d}x$. □

9j Examples: Gamma function; Dirichlet formula; n-dimensional ball

The Euler Gamma function

9j1 Definition.

$$\Gamma(s) = \int_0^\infty t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t \quad \text{for } s > 0 \, .$$

It can be shown to be a continuous function on $(0, \infty)$.[1] Integration by parts gives $\Gamma(s+1) = s\Gamma(s)$. Thus, $\Gamma(n) = (n-1)!$ (by induction, starting with $\Gamma(1) = 1$). Also,

$$\Gamma(\tfrac12) = \int_0^\infty t^{-1/2} \mathrm{e}^{-t} \, \mathrm{d}t = 2 \int_0^\infty \mathrm{e}^{-x^2} \, \mathrm{d}x = \sqrt{\pi} \, .$$

9j2 Exercise. Find the limits $\lim_{s \to 0} s\Gamma(s)$ and $\lim_{s \to 0} \frac{\Gamma(\alpha s)}{\Gamma(s)}$.

There are two remarkable properties of the Γ-function mentioned here without proof. The first one is the identity

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}}$$

that extends the Γ-function to the negative non-integer values of s. The second one is the celebrated Stirling's asymptotic formula

$$\boxed{\Gamma(s) = \sqrt{2\pi} \, s^{s-1/2} \mathrm{e}^{-s} \mathrm{e}^{\theta(s)} \quad \text{for some } \theta(s) \in \Big(0, \frac{1}{12s} \Big)}$$

The Gamma function is very useful in computation of integrals.

[1] Can you do it via Theorem 9h7?

9j3 Claim.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{for } \alpha,\beta>0\,.$$

The left hand side is called the Beta function and denoted by by $\mathrm{B}(\alpha,\beta)$.

Proof. $\Gamma(\alpha+\beta)\mathrm{B}(\alpha,\beta) = \int_0^\infty u^{\alpha+\beta-1}\mathrm{e}^{-u}\,\mathrm{d}u \cdot \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$; we turn it into a two-dimensional integral and change the variables u,x to t_1,t_2 as follows:

$$\begin{cases} t_1 &= ux \\ t_2 &= u(1-x) \end{cases} \qquad \begin{cases} u = t_1+t_2 \\ x = \frac{t_1}{t_1+t_2} \end{cases}$$

This is a diffeomorphism between the first quadrant $t_1,t_2>0$ and the semi-strip $u>0, 0<x<1$. The Jacobian equals $\left|\frac{\partial(t_1,t_2)}{\partial(u,x)}\right| = \left|\begin{smallmatrix} x & u \\ 1-x & -u \end{smallmatrix}\right| = -ux - u + ux = -u$. We obtain

$$\Gamma(\alpha+\beta)B(\alpha,\beta) =$$
$$= \int_0^\infty\int_0^\infty (t_1+t_2)^{\alpha+\beta-1}\mathrm{e}^{-(t_1+t_2)}\Big(\frac{t_1}{t_1+t_2}\Big)^{\alpha-1}\Big(\frac{t_2}{t_1+t_2}\Big)^{\beta-1}\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1+t_2} = \Gamma(\alpha)\Gamma(\beta)\,.$$

□

9j4 Example. Consider the integral

$$\int_0^{\pi/2} \sin^{\alpha-1}\theta\cos^{\beta-1}\theta\,\mathrm{d}\theta\,.$$

Rewriting it in the form

$$\int_0^{\pi/2} \big(\sin^2\theta\big)^{\alpha/2-1}\big(\cos^2\theta\big)^{\beta/2-1}\sin\theta\cos\theta\,\mathrm{d}\theta$$

and changing the variable,

$$\sin^2\theta = x, \quad \mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta\,,$$

we get

$$\frac12\mathrm{B}\Big(\frac{\alpha}{2},\frac{\beta}{2}\Big) = \frac12\cdot\frac{\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)}\,.$$

A special case of this formula says that

$$\int_0^{\pi/2}\sin^{\alpha-1}\theta\,\mathrm{d}\theta = \int_0^{\pi/2}\cos^{\alpha-1}\theta\,\mathrm{d}\theta = \frac12\cdot\frac{\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac12\big)}{\Gamma\big(\frac{\alpha+1}{2}\big)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\big(\frac{\alpha}{2}\big)}{\Gamma\big(\frac{\alpha+1}{2}\big)}\,.$$

9j5 Exercise. Check that $B(x,x) = 2^{1-2x} B(x, \frac12)$.
Hint: $\int_0^{\pi/2} \bigl(\frac{2\sin\theta\cos\theta}{2} \bigr)^{2x-1} \, \mathrm{d}\theta$.

9j6 Exercise. Check the *duplication formula*:
$$\boxed{\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \, \Gamma\Bigl(x + \frac12 \Bigr)}\,.$$
Hint: use 9j5.

9j7 Exercise. Calculate $\int_0^1 x^4 \sqrt{1-x^2} \, \mathrm{d}x$.
Answer: $\frac{\pi}{32}$.

9j8 Exercise. Calculate $\int_0^\infty x^m \mathrm{e}^{-x^n} \, \mathrm{d}x$.
Answer: $\frac1n \Gamma\bigl(\frac{m+1}{n} \bigr)$.

9j9 Exercise. Calculate $\int_0^1 x^m (\ln x)^n \, \mathrm{d}x$.
Answer: $\frac{(-1)^n n!}{(m+1)^{n+1}}$.

9j10 Exercise. Calculate $\int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{\cos x}}$.
Answer: $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

9j11 Exercise. Check that $\Gamma(p)\Gamma(1-p) = \int_0^\infty \frac{x^{p-1}}{1+x} \, \mathrm{d}x$.
Hint: change x to t via $(1+x)(1-t) = 1$.

We mention without proof another useful formula
$$\int_0^\infty \frac{x^{p-1}}{1+x} \, \mathrm{d}x = \frac{\pi}{\sin \pi p} \qquad \text{for } 0 < p < 1\,.$$
There is a simple proof that that uses the residues theorem from the complex analysis course. This formula yields that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$.

The Dirichlet formula

9j12 Proposition.
$$\underset{\substack{x_1,\dots x_n \ge 0, \\ x_1+\dots+x_n \le 1}}{\int \cdots \int} x_1^{p_1-1} \dots x_n^{p_n-1} \, \mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(p_1) \dots \Gamma(p_n)}{\Gamma(p_1 + \dots + p_n + 1)}$$
for $p_1, \dots p_n > 0$.

Proof. Induction in the dimension n. For $n = 1$ the formula is obvious:

$$\int_0^1 x_1^{p_1-1}\,\mathrm{d}x_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1+1)}\,.$$

Now denote the n-dimensional integral by I_n and assume that the result is valid for $n-1$. Then

$$I_n = \int_0^1 x_n^{p_n-1}\,\mathrm{d}x_n \underset{\substack{x_1,\dots,x_{n-1}\ge 0\\ x_1+\dots+x_{n-1}\le 1-x_n}}{\int\cdots\int} x_1^{p_1-1}\dots x_{n-1}^{p_{n-1}-1}\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1}\,.$$

In order to compute the inner integral we change the variables: $x_1 = (1-x_n)\xi_1$, $\dots$, $x_{n-1} = (1-x_n)\xi_{n-1}$. The inner integral becomes

$$(1-x_n)^{n-1+(p_1-1)+\dots+(p_{n-1}-1)} \underset{\substack{\xi_1,\dots\xi_{n-1}\ge 0\\ \xi_1+\dots+\xi_{n-1}\le 1}}{\int\cdots\int} \xi_1^{p_1-1}\dots\xi_{n-1}^{p_{n-1}-1}\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_{n-1} =$$

$$= (1-x_n)^{p_1+\dots+p_{n-1}}\,I_{n-1}\,.$$

Thus,

$$I_n = I_{n-1}\int_0^1 (1-x_n)^{p_1+\,\dots\,+p_{n-1}} x_n^{p_n-1}\,\mathrm{d}x_n =$$

$$= \frac{\Gamma(p_1)\dots\Gamma(p_{n-1})}{\Gamma(p_1+\dots+p_{n-1}+1)}\cdot\frac{\Gamma(p_1+\dots+p_{n-1}+1)\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)} = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}\,.$$

□

There is a seemingly more general formula,

$$\underset{\substack{x_1,\dots,x_n\ge 0,\\ x_1^{\gamma_1}+\dots+x_n^{\gamma_n}\le 1}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{\gamma_1\dots\gamma_n}\cdot\frac{\Gamma\big(\frac{p_1}{\gamma_1}\big)\dots\Gamma\big(\frac{p_n}{\gamma_n}\big)}{\Gamma\big(\frac{p_1}{\gamma_1}+\dots+\frac{p_n}{\gamma_n}+1\big)}\,,$$

easily obtained from the previous one by the change of variables $y_j = x_j^{\gamma_j}$.

A special case: $p_1 = \dots = p_n = 1$, $\gamma_1 = \dots = \gamma_n = p$;

$$\underset{\substack{x_1,\dots,x_n\ge 0\\ x_1^p+\dots+x_n^p\le 1}}{\int\cdots\int} \mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma^n\big(\frac{1}{p}\big)}{p^n\Gamma\big(\frac{n}{p}+1\big)}\,.$$

We've found the volume of the unit ball in the metric l_p:

$$\boxed{v_n\big(B_p(1)\big) = \frac{2^n\Gamma^n\big(\frac1p\big)}{p^n\Gamma\big(\frac np + 1\big)}\,.}$$

If $p = 2$, the formula gives us the volume of the standard unit ball:

$$\boxed{v_n = v_n(B_2(1)) = \frac{2\pi^{n/2}}{n\Gamma\big(\frac n2\big)}\,.}$$

We also see that the volume of the unit ball in the l_1-metric equals $\frac{2^n}{n!}$.

Question: what does the formula give in the $p \to \infty$ limit?

9j13 Exercise. Show that

$$\underset{\substack{x_1+\dots+x_n\le 1\\ x_1,\dots,x_n\ge 0}}{\int\cdots\int} \varphi(x_1+\dots+x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{(n-1)!}\int_0^1 \varphi(s)s^{n-1}\,\mathrm{d}s$$

for every "good" function $\varphi : [0,1] \to \mathbb{R}$ and, more generally,

$$\underset{\substack{x_1+\dots+x_n\le 1\\ x_1,\dots,x_n\ge 0}}{\int\cdots\int} \varphi(x_1+\dots+x_n)x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n =$$

$$= \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n)}\int_0^1 \varphi(u)u^{p_1+\dots p_n-1}\,\mathrm{d}u\,.$$

Hint: consider

$$\int_0^1 \mathrm{d}s\,\varphi'(s) \underset{\substack{x_1+\dots+x_n\le s\\ x_1,\dots,x_n\ge 0}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

9k Oscillation function

Let $f : \mathbb{R}^n \to \mathbb{R}$, and $X = \{(x,t) : t < f(x)\} \subset \mathbb{R}^{n+1}$. Then the interior of X is

$$X^\circ = \{(x,t) : t < f_*(x)\}$$

where $f_* : \mathbb{R}^n \to [-\infty,\infty)$ is defined by[1]

$$f_*(x_0) = \liminf_{x\to x_0} f(x) = \sup_{\delta>0}\inf_{|x-x_0|\le\delta} f(x)\,.$$

[1]Here, "$x \to x_0$" includes the case $x = x_0$.

The proof is simple: $(x_0,t_0) \in X^\circ \iff \exists \delta,\varepsilon>0\ \forall x,t\ \big(|x-x_0|\le\delta \wedge |t-t_0|\le\varepsilon \implies t<f(x)\big) \iff \exists\delta>0\ t_0<\inf_{|x-x_0|\le\delta} f(x) \iff t_0<f_*(x_0)$.

Similarly, for $Y=\{(x,t): t>f(x)\}$ we have

$$Y^\circ = \{(x,t): t>f^*(x)\} \quad \text{where} \quad f^*(x_0) = \limsup_{x\to x_0} f(x) = \inf_{\delta>0} \sup_{|x-x_0|\le\delta} f(x) \in (-\infty,+\infty] \,.$$

The set

$$\Gamma_f = \mathbb{R}^{n+1} \setminus (X^\circ \uplus Y^\circ) = \{(x,t): f_*(x)\le t\le f^*(x)\}$$

is closed, and contains the graph of f (as well as its closure). An example: $f=1\!\!1_{[0,\infty)}:\mathbb{R}\to\mathbb{R}$; Γ consists of the graph of f and a vertical segment $\{0\}\times[0,1]$.

9k1 Definition. The oscillation function $\operatorname{Osc}_f:\mathbb{R}^n\to[0,+\infty]$ is defined by

$$\operatorname{Osc}_f(x_0) = f^*(x_0)-f_*(x_0) = \limsup_{x\to x_0} f(x) - \liminf_{x\to x_0} f(x) = \\ = \inf_{\delta>0} \sup_{|x_1-x_0|\le\delta, |x_2-x_0|\le\delta} |f(x_1)-f(x_2)| \,.$$

Clearly, f is continuous at x if and only if $\operatorname{Osc}_f(x)=0$.

9k2 Theorem. The following three conditions on a bounded function $f: B\to\mathbb{R}$ on a box $B\subset\mathbb{R}^n$ are equivalent:

(a) f is integrable;

(b) ${}^*\!\!\int_B \operatorname{Osc}_f = 0$;

(c) for every $\varepsilon>0$ the set $\{x\in B: \operatorname{Osc}_f(x)\ge\varepsilon\}$ is of volume zero.

9k3 Proposition. If a function f is bounded on a box $B\subset\mathbb{R}^n$ then

$$\sideset{^*}{}\int_B f - \sideset{_*}{}\int_B f = \sideset{^*}{}\int_B \operatorname{Osc}_f \,.$$

9k4 Lemma. $v_*(G_1\uplus G_2) = v_*(G_1)+v_*(G_2)$ whenever $G_1,G_2\subset\mathbb{R}^n$ are disjoint open sets.

Proof. We approximate $G_1\uplus G_2$ from within by a pixelated set and note that each pixel, being connected, is contained either in G_1 or G_2. □

Proof of Prop. 9k3. We take M such that $\sup_B|f|<M$, introduce sets

$$X = \{(x,t): x\in B, -M<t<f(x)\}\,,$$
$$\Gamma = \{(x,t): x\in B^\circ, f_*(x)\le t\le f^*(x)\} = \Gamma_f\cap(B^\circ\times\mathbb{R})\,,$$
$$Y = \{(x,t): x\in B, f(x)<t<M\}$$

and note that

$$X^\circ = \{(x,t) : x \in B^\circ, -M < t < f_*(x)\}\,,$$
$$Y^\circ = \{(x,t) : x \in B^\circ, f^*(x) < t < M\}\,,$$
$$B^\circ \times (-M,M) = X^\circ \uplus \Gamma \uplus Y^\circ\,.$$

By 9c6 and 9k4,

$$v_*(X^\circ) + v^*(\Gamma) + v_*(Y^\circ) = 2Mv(B)\,.$$

It is sufficient to prove that

(a) $$v^*(\Gamma) = {}^*\!\!\int_B \operatorname{Osc}_f\,,$$

(b) $$v_*(X) = {}_*\!\!\int_B f + Mv(B)\,,$$

(c) $$v_*(Y) = Mv(B) - {}^*\!\!\int_B f\,,$$

since $v_*(X) = v_*(X^\circ)$ and $v_*(Y) = v_*(Y^\circ)$ by 6k1.

We have $\Gamma_x = [f_*(x), f^*(x)]$, thus (a) follows from 9i9.

By 9h1, ${}_*\!\int_B (f + M) = v_*\big(X + (0,M)\big)$, which implies (b) via 9c4. Similarly, ${}_*\!\int_B (M - f) = v_*\big(-Y + (0,M)\big)$ implies (c). □

9k5 Lemma. The following two conditions on a bounded function $f : B \to \mathbb{R}$ on a box $B \subset \mathbb{R}^n$ are equivalent:

(a) ${}^*\!\int_B |f| = 0$;

(b) for every $\varepsilon > 0$ the set $\{x \in B : |f(x)| \ge \varepsilon\}$ is of volume zero.

Proof. Denote $A = \{x : |f(x)| \ge \varepsilon\}$.

(a)$\Longrightarrow$(b): $\varepsilon v^*(A) = {}^*\!\int_B \varepsilon 1\!\!1_A \le {}^*\!\int_B |f| = 0$, since $\varepsilon 1\!\!1_A \le |f|$.

(b)$\Longrightarrow$(a): ${}^*\!\int_B |f| = {}^*\!\int_{B\setminus A} |f| \le {}^*\!\int_{B\setminus A} \varepsilon \le \varepsilon v(B)$ for all $\varepsilon > 0$. □

Proof of Theorem 9k2. By 9k3, (a)$\Longleftrightarrow$(b); by 9k5, (b)$\Longleftrightarrow$(c). □

9l On Lebesgue's theory

Here is a bridge from Jordan measure and Riemann integral to Lebesgue measure and Lebesgue integral.

For a set $X \subset \mathbb{R}^n$,

$*$ the *inner Lebesgue measure*

$$m_*(X) = \sup_{\text{compact } K \subset X} v^*(K)\,;$$

* the *outer Lebesgue measure*

$$m^*(X) = \inf_{\text{open } G \supset X} v_*(G) \, ;$$

* X is called *Lebesgue measurable* iff $m_*(X_r) = m^*(X_r)$ for all r; in this case its *Lebesgue measure*

$$m(X) = \lim_{r\to\infty} m_*(X_r) = m_*(X) = \lim_{r\to\infty} m^*(X_r) = m^*(X) \in [0,\infty]$$

(here $X_r = \{x \in X : |x| \le r\}$, as in Sect. 9b).

Note the "bidirectional" limiting procedure:

$$m_*(X) = \sup_{K\subset X} \inf_{E\supset K} v(E) \, , \quad m^*(X) = \inf_{G\supset X} \sup_{E\subset G} v(E) \, ,$$

where E runs over Jordan (or just pixelated) sets, K compact and G open.

A set of Lebesgue measure zero is called *null* (or *negligible*) set. "Almost all" means "all except for a null set".

For a function $f : \mathbb{R}^n \to \mathbb{R}$,

* f is called *Lebesgue measurable* iff the set $\{(x,t) : t < f(x)\} \subset \mathbb{R}^{n+1}$ is Lebesgue measurable;
* in this case the *Lebesgue integral*

$$\int_{\mathbb{R}} f = m\big(\{(x,t) : 0 < t < f(x)\}\big) - m\big(\{(x,t) : f(x) < t < 0\}\big) \, ;$$

four cases appear, similarly to 9d4: real (integrable), $+\infty$, $-\infty$ and $\infty - \infty$.

Here are some facts (not to be proved or used in this course).

* Every locally Jordan set A is Lebesgue measurable; $m(A) = v_*(A)$. Every Jordan measurable function is Lebesgue measurable, with the same integral.
* Lebesgue measurable sets are a σ-algebra (in other words, σ-field) of sets (in $\mathbb{R}^n$). That is, $\emptyset$, $\mathbb{R}^n \setminus A$, $A_1 \cup A_2 \cup \dots$ (and therefore also $\mathbb{R}^n$ and $A_1 \cap A_2 \cap \dots$) are Lebesgue measurable whenever $A, A_1, A_2, \dots$ are. This σ-algebra contains all open sets (as well as all closed sets).

Note that $m(G) = v_*(G)$, $m(K) = v^*(K)$ for open G and compact K.

* *(σ-additivity)* $m(A_1 \uplus A_2 \uplus \dots) = m(A_1) + m(A_2) + \dots$ whenever $A_1, A_2, \dots$ are disjoint Lebesgue measurable sets.
* *(Monotone convergence for sets)* Let sets A_i be Lebesgue measurable. If $A_i \uparrow A$ then $m(A_i) \uparrow m(A)$; if $A_i \downarrow A$ and $m(A_1) < \infty$ then $m(A_i) \downarrow m(A)$.

∗ All locally volume zero sets are null sets. A countable union of null sets is a null set.

Now we are in position to reformulate Theorem 9k2:

∗ *(Lebesgue's criterion of Riemann integrability)* A bounded function with bounded support is Riemann integrable if and only if it is continuous almost everywhere. A function is Jordan measurable if and only if it is continuous almost everywhere.

∗ *(Monotone convergence for functions)* Let functions f_i be Lebesgue measurable. If $0 \le f_i \uparrow f$ then $\int f_i \uparrow \int f$. If $f_i \downarrow f \ge 0$ and $\int f_1 < \infty$ then $\int f_i \downarrow \int f$.

∗ *(Tonelli: Iterated Lebesgue integral for positive functions)* If $f : \mathbb{R}^{n+m} \to [0,\infty)$ is Lebesgue measurable then f_x is Lebesgue measurable for almost all x, the function $x \mapsto \int f_x$ is Lebesgue measurable,[1] and $\int_{\mathbb{R}^{n+m}} f = \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^m} \mathrm{d}y\, f(x,y)$.

∗ *(Fubini: Iterated Lebesgue integral for integrable functions)* If $f : \mathbb{R}^{n+m} \to [0,\infty)$ is integrable then f_x is integrable for almost all x, the function $x \mapsto \int f_x$ is integrable and $\int_{\mathbb{R}^{n+m}} f = \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^m} \mathrm{d}y\, f(x,y)$.

Note that all lower integrals in Theorem 9i2 are equal to Lebesgue integrals.

∗ *(Change of variables)* The same as Theorem 9f5, with "Lebesgue measurable" in place of "Jordan measurable".

∗ *(Dominated convergence)* If $f_i : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. $f_i \to f$ pointwise, and $\int \sup_i |f_i| < \infty$ then $\int f_i \to \int f$.

The choice axiom leads to a proof of existence of sets and functions that fail to be Lebesgue measurable; but not to specific[2] examples of such monsters.

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[1]No matter how it is defined on the null set...

[2]I mean, definable without parameters.