Solutions to selected exercises

3e3 Exercise. Consider the set $U \subset \mathbb{R}^n$ of all $(a_0, \dots, a_{n-1})$ such that the polynomial

$$t \mapsto t^n + a_{n-1}t^{n-1} + \dots + a_0$$

has n pairwise distinct real roots.

(a) Prove that U is open.

(b) Define $\psi : U \to \mathbb{R}^n$ by $\psi(a_0, \dots, a_{n-1}) = (t_1, \dots, t_n)$ where $t_1 < \dots < t_n$ are the roots of the polynomial. Prove that ψ is a homeomorphism $U \to V$ where $V = \{(t_1, \dots, t_n) : t_1 < \dots < t_n\}$.[1]

Solution.

The set V is open (evidently). We consider a mapping $\varphi : V \to U$ defined by

$$\varphi(t_1, \dots, t_n) = (a_0, \dots, a_{n-1}) \quad \text{whenever} \quad \forall x \;\; (t - t_1) \dots (t - t_n) = t^n + a_{n-1}t^{n-1} + \dots + a_0 \,.$$

It is continuously differentiable, since each a_k is a polynomial function of $t_1, \dots, t_n$. We note that $\varphi(t_1, \dots, t_n) = (a_0, \dots, a_{n-1})$ if and only if $\psi(a_0, \dots, a_{n-1}) = (t_1, \dots, t_n)$; that is, $\varphi^{-1} = \psi$.

According to the hint we use 2e9(b). Treating $a_0, \dots, a_{n-1}$ as coordinates in the space S_n of 2e9 we have

$$\text{the operator } (D\varphi)_{(t_1,\dots,t_n)} \text{ is invertible}$$

for all $(t_1, \dots, t_n) \in V$.

By Theorem 4c1 applied to φ near an arbitrary point $v \in V$, the set $\varphi(V) = U$ is a neighborhood of $\varphi(v) = u$, and the mapping $\varphi^{-1} = \psi$ is continuous at u.

This holds for every $u \in U$; thus U is open, and ψ is continuous. Taking into account continuity of $\psi^{-1} = \varphi$ we see that ψ is a homeomorphism. □

3i1 Exercise. Let A be an invertible linear operator. Find $\|A^{-1}\|$.

Solution. Using 1e1 and taking $x = A^{-1}y$ we have

$$\|A^{-1}\| = \sup_{y \in \mathbb{R}^n, y \ne 0} \frac{|A^{-1}y|}{|y|} = \sup_{x \in \mathbb{R}^n, x \ne 0} \frac{|x|}{|Ax|} \,;$$

$$\frac{1}{\|A^{-1}\|} = \inf_{x \in \mathbb{R}^n, x \ne 0} \frac{|Ax|}{|x|} = \min_{|x|=1} |Ax| \,.$$

[1] Hint: use 2e9(b).

It was shown before Exercise 3i1 that $\max_{|x|=1} |Ax|^2$ is the maximal eigenvalue $\lambda_{\max}$ of A^*A. Similarly, $\min_{|x|=1} |Ax|^2$ is the minimal eigenvalue $\lambda_{\min}$ of A^*A. Thus, $\|A^{-1}\|^2 = 1/\lambda_{\min}$. □

4c9 Exercise. (a) Let $f : U \to V$ be as in Theorem 4c5 and in addition $f \in C^2(U)$ (recall Sect. 2g). Prove that $f^{-1} \in C^2(V)$.[1]

(b) The same for $C^k(\dots)$ where $k = 3, 4, \dots$

Solution. (a) Denote $g = f^{-1}$. The mapping $y \mapsto (Dg)_y = \big((Df)_{g(y)}\big)^{-1}$ is the composition of three mappings. First, $y \mapsto g(y) = x$. Second, $x \mapsto (Df)_x = A$. Third, $A \mapsto A^{-1}$. The first mapping $g : V \to U$ is continuously differentiable by Theorem 4c5. The second mapping is continuously differentiable on U since $f \in C^2(U)$. The third mapping is continuously differentiable, see 4c8. Therefore their composition $y \mapsto (Dg)_y$ is continuously differentiable, which means that $g \in C^2(V)$.

(b) Induction in k. By the induction hypothesis, the first mapping g belongs to $C^{k-1}(V)$. The second mapping $x \mapsto (Df)_x$ belongs to $C^{k-1}(U)$ since $f \in C^k(U)$. The third mapping $A \mapsto A^{-1}$ belongs to C^m for all m, since elements of A^{-1} are just rational functions (that is, fractions of polynomials) of the elements of A (as noted in 4c8). Therefore their composition belongs to C^{k-1}, which means that $g \in C^k(V)$. □

[1] Hint: $(Dg)_y = ((Df)_{g(y)})^{-1}$ where $g = f^{-1}$.