Solutions to selected exercises

6#1(b) Exercise. Differentiate $S : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$, $S(A) = A^t A$.

Solution. $S(A+H) = (A+H)^t(A+H) = A^tA + A^tH + H^tA + H^tH = S(A) + (A^tH + H^tA) + o(\|H\|)$; $(DS)_A(H) = A^tH + H^tA = A^tH + (A^tH)^t$. $\square$

6#1(d) Exercise. Differentiate $P : M_{n,n}(\mathbb{R}) \to P_n$, $P(A)(x) = \det(xI - A)$, at the point I.

Solution. By 2e7(b), $(D\det)_I = \operatorname{tr}$, that is, $\det(I+H) = 1 + \operatorname{tr}(H) + o(\|H\|)$. Thus, for $x \ne 1$,

$$P(I+H)(x) = \det\big(xI - (I+H)\big) = \det\big((x-1)I - H\big) =$$
$$= (x-1)^n \det\Big(I - \frac{1}{x-1}H\Big) = (x-1)^n\bigg(1 + \operatorname{tr}\Big(-\frac{1}{x-1}H\Big) + o(\|H\|)\bigg) =$$
$$= (x-1)^n - (x-1)^{n-1}\operatorname{tr} H + o(\|H\|)\,;$$

finally, $(DP)_I(H)(x) = -(x-1)^{n-1}\operatorname{tr} H$. $\square$

6#3 Exercise. Define a mapping $f : U \to M_{d,d}(\mathbb{R})$, where $U = \{A \in M_{d,d}(\mathbb{R}) : \|A\| < 1\}$ (the operator norm being used), by

$$f(A) = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{A^k}{k} \quad \text{for } \|A\| < 1$$

(it is in fact $\log(I+A)$). Prove that

(a) the series converges;

(b) f is continuously differentiable;

(c) f is open on some neighborhood of 0;

**(d) $\log(\exp(A)) = A$ for all A in some neighborhood of 0.

Solution. (a) Partial sums are a Cauchy sequence, since

$$\Big\| \sum_{k=m}^{m+n} (-1)^{k+1}\frac{A^k}{k} \Big\| \le \sum_{k=m}^{m+n} \Big\| (-1)^{k+1}\frac{A^k}{k} \Big\| =$$
$$= \sum_{k=m}^{m+n} \frac{1}{k}\|A^k\| \le \sum_{k=m}^{m+n} \frac{1}{k}\|A\|^k \le \sum_{k=m}^{\infty} \|A\|^k = \frac{\|A\|^m}{1-\|A\|} \to 0$$

as $m \to \infty$.

(b) First, consider (for arbitrary k) a mapping $g_k : M_{d,d} \to M_{d,d}$,

$$g_k(A) = A^k\,.$$

We have

$$g_k(A+H) = (A+H)^k = \sum_{i_1,\dots,i_k=0,1} A^{1-i_1}H^{i_1}\dots A^{1-i_k}H^{i_k} =$$

$$= \underbrace{A^k}_{g_k(A)} + \underbrace{A^{k-1}H + A^{k-2}HA + \dots + HA^{k-1}}_{(Dg_k)_A(H)} + \underbrace{\sum_{i_1+\dots+i_k\ge 2} A^{1-i_1}H^{i_1}\dots A^{1-i_k}H^{i_k}}_{o(\|H\|)} ;$$

$$\|g_k(A+H) - g_k(A) - (Dg_k)_A(H)\| \le \sum_{i_1+\dots+i_k\ge 2} \|A^{1-i_1}H^{i_1}\dots A^{1-i_k}H^{i_k}\| \le$$

$$\le \sum_{i_1+\dots+i_k\ge 2} \|A\|^{1-i_1}\|H\|^{i_1}\dots\|A\|^{1-i_k}\|H\|^{i_k} =$$

$$= \big(\|A\|+\|H\|\big)^k - \|A\|^k - k\|A\|^{k-1}\|H\| \le \frac12 k(k-1)(\|A\|+\|H\|)^{k-2}\|H\|^2 .$$

The series $\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} g_k(A) = f(A)$ converges by (a); also the series $\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} g_k(A+H) = f(A+H)$ converges when $\|H\| < 1 - \|A\|$; and the series

$$\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\big(g_k(A+H) - g_k(A) - (Dg_k)_A(H)\big)$$

converges for these H, since

$$\sum_{k=1}^\infty \Big\|\frac{(-1)^{k+1}}{k}\big(g_k(A+H) - g_k(A) - (Dg_k)_A(H)\Big\| \le$$

$$\sum_{k=1}^\infty \frac1k \cdot \frac12 k(k-1)(\|A\|+\|H\|)^{k-2}\|H\|^2 < \infty .$$

Therefore the series $\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}(Dg_k)_A(H)$ converges, and

$$\Big\| f(A+H) - f(A) - \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}(Dg_k)_A(H)\Big\| \le$$

$$\le \sum_{k=1}^\infty \frac{k-1}{2}(\|A\|+\|H\|)^{k-2}\|H\|^2 = o(\|H\|) .$$

We see that

$$(Df)_A = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}(Dg_k)_A , \quad (Dg_k)_A(H) = A^{k-1}H + A^{k-2}HA + \dots + HA^{k-1} .$$

Each Dg_k is evidently continuous, and the series converges uniformly on $\{A : \|A\| \le 1-\varepsilon\}$ for every $\varepsilon > 0$, therefore Df is continuous.

(c) Clearly, $(Dg_1)_0 = \mathrm{id}$ and $(Dg_k)_0 = 0$ for $k > 1$; thus $(Df)_0 = \mathrm{id}$. It follows that f is open on some neighborhood of 0 (see Theorem 4c1 and Exercise 3b3).

(**d) *(Sketch only)* First, for arbitrary polynomials f and g,

$$g(f(A)) = (g \circ f)(A)$$

(this algebraic identity follows from definitions). The problem is that our functions f, g, $f(x) = \mathrm{e}^x - 1$ and $g(x) = \log(1+x)$, are not polynomials (but power series).

Second, the Jordan normal form[1] reduces the general case to the special case

$$A = \lambda I + T\,, \quad T^d = 0\,.$$

For arbitrary polynomial f,

$$f(A) = \sum_{k=0}^{d-1} \frac{1}{k!} f^{(k)}(\lambda) T^k =$$
$$= f(\lambda)I + f'(\lambda)T + \frac{1}{2} f''(\lambda)T^2 + \dots + \frac{1}{(d-1)!} f^{(d-1)}(\lambda) T^{d-1}\,.$$

It follows that the same equality holds whenever f is a power series whose radius of convergence exceeds $|\lambda|$. Moreover, if f_k are polynomials such that $f_k(\lambda) \to f(\lambda)$, $f'_k(\lambda) \to f'(\lambda)$, $\dots$, $f_k^{(d-1)}(\lambda) \to f^{(d-1)}(\lambda)$ as $k \to \infty$, then $f_k(A) \to f(A)$.

Third, let f_n be the n-th Taylor sum for f, $f(x) = \mathrm{e}^x - 1$, and similarly g_n for g, $g(x) = \log(1+x)$. It appears that $f_n \to f$, $g_n \to g$ and $g_n \circ f_n \to g \circ f$, the convergence being the locally uniform (near 0) convergence *of functions and all derivatives.* □

8#2 Exercise. Prove that the mapping[2]

$$S : \mathbb{R}_+ \times (0,\pi)^{n-2} \times \mathbb{R} \to \mathbb{R}^n \setminus \operatorname{Span}\{e_3, \dots, e_n\}$$

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[1] See Wikipedia, articles "Jordan normal form" and "Logarithm of a matrix" (item "The logarithm of a non-diagonalizable matrix").

[2] The original formulation contains $\cup_{j=3}^n \operatorname{Span}\{e_j\}$ rather than $\operatorname{Span}\{e_3, \dots, e_n\}$; this is a mistake, sorry.

defined by equations

$$\begin{aligned}
x_n &= r\cos\theta_{n-2}\\
x_{n-1} &= r\sin\theta_{n-2}\cos\theta_{n-3}\\
&\dots\\
x_3 &= r\sin\theta_{n-2}\sin\theta_{n-3}\dots\sin\theta_2\cos\theta_1\\
x_2 &= r\sin\theta_{n-2}\sin\theta_{n-3}\dots\sin\theta_2\sin\theta_1\cos\varphi\\
x_1 &= r\sin\theta_{n-2}\sin\theta_{n-3}\dots\sin\theta_2\sin\theta_1\sin\varphi
\end{aligned}$$

is locally invertible, and satisfies[1]

$$\det(DS) = r^{n-1}\prod_{j=1}^{n-2}\sin^j\theta_j\,.$$

Solution. First we prove that $S(U) = V$ where $U = (0,\infty)\times(0,\pi)^{n-2}\times\mathbb{R}$ and $V = \mathbb{R}^n \setminus \operatorname{Span}\{e_3,\dots,e_n\} = \{(x_1,\dots,x_n) : x_1^2 + x_2^2 > 0\}$. We introduce $r_k = \sqrt{x_1^2 + \dots + x_k^2}$ and note that

$$\begin{gathered}
r_k = r\sin\theta_{n-2}\dots\sin\theta_{k-1} \quad \text{for } k = 2,\dots,n\,,\\
x_k = r_k\cos\theta_{k-2} \quad \text{for } k = 3,\dots,n\,.
\end{gathered}$$

Thus, $x_1^2 + x_2^2 = r_2^2 = (r\sin\theta_{n-2}\dots\sin\theta_1)^2 > 0$ (since $\theta_1,\dots,\theta_{n-2} \in (0,\pi)$), that is, $S(U) \subset V$.

Given $x \in V$, we take $\theta_{k-2} \in (0,\pi)$ such that $\cos\theta_{k-2} = x_k/r_k$ for $k = 3,\dots,n$, then $\sin\theta_{k-2} = \sqrt{1 - \frac{x_k^2}{r_k^2}} = \sqrt{\frac{r_k^2 - x_k^2}{r_k^2}} = r_{k-1}/r_k$ for $k = 3,\dots,n$, therefore $r_k = r\sin\theta_{n-2}\dots\sin\theta_{k-1}$ for $k = 2,\dots,n$, and $x_k = r_k\cos\theta_{k-2} = r\sin\theta_{n-2}\dots\sin\theta_{k-1}\cos\theta_{k-2}$ for $k = 3,\dots,n$. We take some (non-unique) $\varphi \in \mathbb{R}$ such that $\cos\varphi = x_2/r_2$ and $\sin\varphi = x_1/r_2$, then $x_2 = r_2\cos\varphi = r\sin\theta_{n-2}\dots\sin\theta_1\cos\varphi$ and $x_1 = r_2\sin\varphi = r\sin\theta_{n-2}\dots\sin\theta_1\sin\varphi$, which shows that $x \in S(U)$. We see that $S(U) = V$.

Second, we find $\det(DS)$. Denoting the matrix DS by A,

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n}\\ \dots & \dots & \dots\\ a_{n,1} & \dots & a_{n,n}\end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_1}{\partial\theta_1} & \dots & \frac{\partial x_1}{\partial\theta_{n-2}}\\ \dots & \dots & \dots & \dots & \dots\\ \frac{\partial x_n}{\partial r} & \frac{\partial x_n}{\partial\varphi} & \frac{\partial x_n}{\partial\theta_1} & \dots & \frac{\partial x_n}{\partial\theta_{n-2}}\end{pmatrix}$$

[1]The original formulation contains $\det(DS)$ rather than $|\det(DS)|$; however, the sign of the determinant depends on the enumeration of the variables.

and the corresponding matrix in dimension $n-1$ by B, we observe that the minor $A_{n,n}$ is propotional to B,

$$A_{n,n} = \sin\theta_{n-2} \cdot B\,, \quad \text{that is, } a_{k,l} = \sin\theta_{n-2} \cdot b_{k,l} \text{ for } k,l = 1,\dots,n-1\,.$$

Therefore $\det A_{n,n} = \sin^{n-1}\theta_{n-2} \cdot \det B$.

We also note that the first and last columns of A are proportional, except for the last element,

$$a_{i,n} = r\frac{\cos\theta_{n-2}}{\sin\theta_{n-2}} a_{i,1} \quad \text{for } i = 1,\dots,n-1\,.$$

Without changing $\det A$ we add the first column multiplied by $\bigl(-r\frac{\cos\theta_{n-2}}{\sin\theta_{n-2}}\bigr)$ to the last column; we get

$$a_{1,n} = \dots = a_{n-1,n} = 0\,,$$

$$a_{n,n} = \frac{\partial x_n}{\partial\theta_{n-2}} - r\frac{\cos\theta_{n-2}}{\sin\theta_{n-2}}\frac{\partial x_n}{\partial r} = -r\sin\theta_{n-2} - r\frac{\cos\theta_{n-2}}{\sin\theta_{n-2}}\cos\theta_{n-2} = -\frac{r}{\sin\theta_{n-2}}\,.$$

Finally,

$$\det A = -\frac{r}{\sin\theta_{n-2}}\det A_{n,n} = -\frac{r}{\sin\theta_{n-2}}\sin^{n-1}\theta_{n-2}\cdot\det B = -r\sin^{n-2}\theta_{n-2}\det B\,.$$

The result follows by induction in n. □

8#4 Exercise. Let $\mathbb{R}^2 \ni (u,v) \mapsto F(u,v) = w \in \mathbb{R}$ be a C^1 mapping, $F(0,0) = 0$,[1] and $a,b \in \mathbb{R}$ satisfy

$$a\frac{\partial F}{\partial u}(0,0) + b\frac{\partial F}{\partial v}(0,0) \ne 0\,.$$

Prove that

(a) equation $F(x-az, y-bz) = 0$ in some neighborhood of $(0,0,0)$ determines z as a C^1 function of x,y;

(b) $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$ in this neighborhood.

Solution. We rewrite the equation as $g(x,y,z) = 0$ where $g \in C^1(\mathbb{R}^3 \to \mathbb{R})$ is defined by $g(x,y,z) = F(x-az, y-bz)$. We have

$$\frac{\partial}{\partial z}g(x,y,z) = -a\frac{\partial F}{\partial u}(x-az,y-bz) - b\frac{\partial F}{\partial v}(x-az,y-bz) \ne 0$$

[1] This condition is forgotten in the original formulation, sorry.

at $(0,0,0)$. By Th. 5c1, near $(0,0,0)$ the equation determines z as a C^1 function of x, y. We differentiate the equality

$$F\big(x - az(x,y), y - bz(x,y)\big) = 0$$

in x:

$$\Big(1 - a\frac{\partial z}{\partial x}\Big)\frac{\partial F}{\partial u} - b\frac{\partial z}{\partial x} \cdot \frac{\partial F}{\partial v} = 0\,; \quad \frac{\partial z}{\partial x} = \frac{\frac{\partial F}{\partial u}}{a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}}\,.$$

Similarly (differentiating in y),

$$\frac{\partial z}{\partial y} = \frac{\frac{\partial F}{\partial v}}{a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}}\,.$$

Thus, $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$. □