# 13 Integration, from local to global



*Single-chart pieces of a manifold are combined via partitions of unity.*

## 13a Partition of unity

**13a1 Definition.** (a) A $k$-form $\omega$ on an $n$-manifold $M \subset \mathbb{R}^N$ is *compactly supported* if there exists a compact set $K \subset M$ that supports $\omega$ in the sense that $\omega(x, h_1, \dots, h_k) = 0$ for all $x \in M \setminus K$ and $h_1, \dots, h_k \in T_x M$.

(b) $\omega$ is a *single-chart form* if there exist a compact set $K \subset M$ that supports $\omega$ and a chart $(G, \psi)$ of $M$ such that $K \subset \psi(G)$.

The same applies to continuous functions on $M$ (they are 0-forms).

Recall that $\int_{(M,\mathcal{O})} \omega$ is defined (in Sect. 12c) whenever $(M, \mathcal{O})$ is an oriented $n$-manifold and $\omega$ a single-chart $n$-form on $M$. The linearity,

$$\int_{(M,\mathcal{O})} (c_1\omega_1 + c_2\omega_2) = c_1 \int_{(M,\mathcal{O})} \omega_1 + c_2 \int_{(M,\mathcal{O})} \omega_2 \,, \tag{13a2}$$

is evident provided that both forms $\omega_1, \omega_2$ have compact supports within the same chart.

Every compact subset of $M$ can be covered by finitely many charts. They overlap. We could try to construct a partition of the compact set into simple-chart sets. But it is better to split $\omega$ into single-chart forms, using the so-called "partition of unity".[1]

**13a3 Lemma.** Let $M \subset \mathbb{R}^N$ be an $n$-manifold and $K \subset M$ a compact set. Then there exist single-chart continuous functions $\rho_1, \dots, \rho_i : M \to [0, 1]$ such that $\rho_1 + \dots + \rho_i = 1$ on $K$.

***Proof.*** For every $x \in K$ the function $f_x : y \mapsto \big(\varepsilon_x - |y - x|\big)^+$ is single-chart if $\varepsilon_x$ is small enough; it is also continuous, and (strictly) positive in the open $\varepsilon_x$-neighborhood of $x$. These neighborhoods are an open covering of $K$; we

[1] For now, partition of unity into continuous functions. Later, partition into $C^1$ functions will be needed. (See 13b11. Ultimately, partitions into $C^\infty$ functions exist, but we do not need them.)

choose a finite subcovering and get single-chart functions $f_1, \dots, f_i : M \to [0, \infty)$ whose sum $f = f_1 + \dots + f_i$ is (strictly) positive on $K$. We take $\varepsilon > 0$ such that $f(\cdot) \ge \varepsilon$ on $K$ and note that functions $\rho_1, \dots, \rho_i : M \to [0, \infty)$ defined by

$$\rho_j(x) = \frac{f_j(x)}{\max(f(x), \varepsilon)}$$

have the required properties. $\square$

It follows that every compactly supported $k$-form on $M$ is the sum of some single-chart $k$-forms,

$$\omega = \omega_1 + \dots + \omega_i \, , \quad \omega_j = \rho_j \omega$$

(that is, $\omega_j(x, h_1, \dots, h_k) = \rho_j(x)\omega(x, h_1, \dots, h_k)$).

For $k = n$ we can define (assuming that $\mathcal{O}$ is an orientation of $M$)

$$\text{(13a4)} \qquad \int_{(M,\mathcal{O})} (\omega_1 + \dots + \omega_i) = \int_{(M,\mathcal{O})} \omega_1 + \dots + \int_{(M,\mathcal{O})} \omega_i$$

if this is correct; that is, we need

$$\text{(13a5)} \qquad \int_{(M,\mathcal{O})} \omega_1 + \dots + \int_{(M,\mathcal{O})} \omega_i = \int_{(M,\mathcal{O})} \tilde\omega_1 + \dots + \int_{(M,\mathcal{O})} \tilde\omega_{\tilde i}$$

whenever $\omega_1 + \dots + \omega_i = \tilde\omega_1 + \dots + \tilde\omega_{\tilde i}$. This equality will be proved after some preparation.

All compactly supported $k$-forms on $M$ are a vector space (infinite-dimensional, of course). Forms compactly supported by a given chart are a vector subspace; and these subspace, together, span the whole space. Therefore

(13a6) if two linear functionals on compactly supported forms are equal on all single-chart forms, then they are equal.

In particular, this applies to continuous functions (0-forms).

Given single-chart $n$-forms $\omega_1, \dots, \omega_i$, we introduce the functional

$$L : f \mapsto \int_{(M,\mathcal{O})} f\omega_1 + \dots + \int_{(M,\mathcal{O})} f\omega_i$$

on compactly supported continuous functions $f : M \to \mathbb{R}$; it is linear, since each $\int_{(M,\mathcal{O})} f\omega_j$ is linear by (13a2). By (13a2) (again),

$$L(f) = \int_{(M,\mathcal{O})} f\omega \quad \text{where } \omega = \omega_1 + \dots + \omega_i$$

for single-chart $f$ (for non-single-chart $f$ the right-hand side is generally not defined yet). Given also $\omega = \tilde\omega_1 + \dots + \tilde\omega_{\tilde i}$, we introduce $\tilde L$ and note that

$$L(f) = \int_{(M,\mathcal{O})} f\omega = \tilde L(f)$$

for single-chart $f$. By (13a6), $L = \tilde L$. Choosing $f$ such that $f(\cdot) = 1$ on the union of supports of $\omega_1, \dots, \omega_i, \tilde\omega_1, \dots, \tilde\omega_{\tilde i}$ we get (13a5).

We see that (13a4) is indeed a correct definition of $\int_{(M,\mathcal{O})} \omega$ whenever $\omega$ is a compactly supported $n$-form on $M$.

Now we can define the $n$-dimensional volume of a compact oriented $n$-manifold $(M, \mathcal{O})$ by

$$v(M,\mathcal{O}) = \int_{(M,\mathcal{O})} \mu_{(M,\mathcal{O})} \in (0,\infty)$$

where $\mu_{(M,\mathcal{O})}$ is the volume form on $(M,\mathcal{O})$. However, the Möbius strip should have an area, too!

We want to define

$$\int_M f = \int_{(G,\psi)} f\mu_{(G,\psi)} = \int_G (f\circ\psi) J_\psi \tag{13a7}$$

for a single-chart $f \in C(M)$; here $(G,\psi)$ is a chart such that $f$ is compactly supported within $\psi(G)$, and $\mu_{(G,\psi)}$ is the volume form on the $n$-manifold $\psi(G)$ (oriented, even if $M$ is non-orientable). To this end we need a counterpart of Prop. 12c3:

$$\int_{(G_1,\psi_1)} f\mu_{(G_1,\psi_1)} = \int_{(G_2,\psi_2)} f\mu_{(G_2,\psi_2)}$$

whenever $(G_1,\psi_1)$, $(G_2,\psi_2)$ are charts such that $\psi_1(G_1) = \psi_2(G_2)$ supports $f$. We do it similarly to the proof of 12c3, but this time we split the relatively open set $\tilde G = \psi_1(G_1) = \psi_2(G_2)$ in two relatively open sets $\tilde G_-, \tilde G_+$ according to the sign of $\det D\varphi$ (having $\psi_2^{-1} = \varphi\circ\psi_1^{-1}$ on $\tilde G$). It remains to take into account that $\mu_{(G_1,\psi_1)} = \mu_{(G_2,\psi_2)}$ on $\tilde G_+$ but $\mu_{(G_1,\psi_1)} = -\mu_{(G_2,\psi_2)}$ on $\tilde G_-$.

We see that (13a7) is indeed a correct definition of $\int_M f$ for a single-chart $f \in C(M)$. Now, similarly to (13a4), we define

$$\int_M f = \int_M f_1 + \dots + \int_M f_i \tag{13a8}$$

whenever $f = f_1 + \dots + f_i$ with single-chart $f_j \in C(M)$.

**13a9 Exercise.** (a) Prove that (13a8) is a correct definition of $\int_M f$ for all compactly supported $f \in C(M)$;

(b) formulate and prove linearity and monotonicity of the integral.

Consider a function $f : M \to \mathbb{R}$ continuous almost everywhere.[1] If $f$ is single-chart, we define

$$\int_M f = \int_G (f \circ \psi) J_\psi = (12c8)$$

for a chart $(G, \psi)$ that supports $f$; by 12c9 the integral does not depend on the chart. But now it is treated as improper, and may converge (then $f$ is called integrable) or diverge. This integral is a linear functional on the vector subspace of integrable functions supported by a given chart. Similarly to (13a5) it extends to a linear functional on compactly supported $f$ (continuous almost everywhere, of course). And then, by exhaustion, we get rid of the compact support.

Accordingly, we have the ($n$-dimensional) Jordan measure on $M$, and sets of volume zero. A single point is of volume zero, of course.

**13a10 Exercise.** (a) Every compact subset of an $n$-manifold in $\mathbb{R}^N$ (for $n < N$) is of ($N$-dimensional) volume zero in $\mathbb{R}^N$.[2]

(b) Let $M$ be an $n$-manifold in $\mathbb{R}^N$; $M_1$ an $n_1$-manifold in $\mathbb{R}^N$; $n_1 < n$; and $M_1 \subset M$. Then every compact subset of $M_1$ is of ($n$-dimensional) volume zero in $M$.

Prove it.[3]

**13a11 Example.** Consider the sphere $M = \{(x, y, z) : x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$ and the 2-chart $(G, \psi)$ of the sphere, called the spherical coordinates:

$$G = (-\pi, \pi) \times (0, \pi) , \quad \psi(\varphi, \theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) .$$

The circle $\{(x, 0, z) : x^2 + z^2 = 1\} \subset M$ is a set of volume zero by 13a10(b). Therefore the semicircle $M \setminus \psi(G) = \{(x, 0, z) : x^2 + z^2 = 1, x \le 0\}$ is of volume zero. Using 12c15 and 12c23 we calculate the area of the sphere; not unexpectedly, we get

$$v(M) = \int_G J_\psi = \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta = 4\pi .$$

**13a12 Exercise.** Prove that the area of the (non-compact) Möbius strip 12c20 is $4\pi r R\big(1 + \mathcal{O}\big(\frac{r^2}{R^2}\big)\big)$.

**13a13 Example** (PRODUCT). Let $M_1$ be an $n_1$-manifold in $\mathbb{R}^{N_1}$ and $M_2$ an $n_2$-manifold in $\mathbb{R}^{N_2}$. Then, using 12c27,

$$v(M_1 \times M_2) = v(M_1)v(M_2) \in (0, \infty] .$$

[1] Recall 12b12.

[2] Why just compact? Wait for 13b8.

[3] Hint: (a) locally, a graph; (b) $\psi^{-1} \circ \psi_1$.

**13a14 Example** (SCALING)**.** Let $M$ be an $n$-manifold in $\mathbb{R}^N$, and $s \in (0,\infty)$. Then, using 12c28,

$$v(sM) = s^n v(M) \in (0,\infty] \,.$$

This is a generalization of the special case $v(sE) = s^n v(E)$ of 6g12. In contrast, we have no such generalization of the more general formula $v(T(E)) = s_1 \dots s_n v(E)$ of 6g12. Indeed, everyone knows the length of a circle, but the length of an ellipse is an elliptic integral![1]

**13a15 Example** (MOTION)**.** Let $M$ be an $n$-manifold in $\mathbb{R}^N$, and $T : \mathbb{R}^N \to \mathbb{R}^N$ an isometric affine mapping. Then, using 12c29,

$$v(T(M)) = v(M) \in [0,\infty] \,.$$

Also,

$$\int_{T(M)} f \circ T^{-1} = \int_M f$$

(in the appropriate sense) for every $f : M \to \mathbb{R}$ continuous almost everywhere. In particular, (a) if $f \circ T^{-1} = f$ then $\int_{T(M)} f = \int_M f$; and (b) if $T(M) = M$ then $\int_M f \circ T^{-1} = \int_M f$.

**13a16 Example** (CYLINDER)**.** Let $M_1, h, M$ be as in 12b22, but with $(a,b) \subset \mathbb{R}$ rather than the whole $\mathbb{R}$; and let $\langle h, \cdot \rangle$ be constant on $M_1$. Then, using 12c30,

$$v(M) = (b-a)|h| v(M_1) \,.$$

**13a17 Example** (CONE)**.** Let $M_1$ and $M$ be as in 12b23, but with $(a,b) \subset (0,\infty)$ rather than the whole $(0,\infty)$; and let $\forall x \in M_1 \;\; |x| = c$. Then, using 12c31,

$$v(M) = \frac{c}{n+1}(b^{n+1} - a^{n+1}) v(M_1) \,.$$

**13a18 Example** (SURFACE OF REVOLUTION OR BODY OF REVOLUTION)**.** Let $M_1, n, M$ be as in 12b24, and $M_1 \subset \mathbb{R}^2 \times \{0\}$. Then, using 12c32,

$$v(M) = 2\pi \int_{M_1} |y| \,.$$

Assuming in addition that $M_1 \subset \mathbb{R} \times (0,\infty) \times \{0\}$ we get the Pappus-Guldin centroid theorem:[2] [3]

[1] Wikipedia:Ellipse#Circumference.

[2] See "Pappus's centroid theorem" in Wikipedia.

[3] Centroid, defined in Sect. 9b (before 9b8), generalizes readily to manifolds.

(for $n = 1$) The surface area of a surface of revolution generated by rotating a plane curve about an external axis on the same plane is equal to the product of the arc length of the curve and the distance traveled by its geometric centroid.

(for $n = 2$) The volume of a solid of revolution generated by rotating a plane figure about an external axis on the same plane is equal to the product of the area of the figure and the distance traveled by its geometric centroid.

**13a19 Exercise.** (a) Find the integral of the function $x \mapsto x_i^2$ over the sphere $x_1^2 + \dots + x_N^2 = 1$ without ANY computation.[1]

(b) Prove that $\frac{v(M_a)}{NV_N} \le \frac{1}{2a^2N}$ (here $M_a$ is the spherical cap as in 12c25).

**13a20 Exercise.** Find the area of the part of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ situated inside the cylinder $x^2 + z^2 = 1$ (that is, satisfying $x^2 + z^2 < 1$).[2]

**13a21 Exercise.** Find (a) the area of the part of the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ situated inside the cylinder $x^2 + y^2 = x$; and (b) the area of the part of the cylinder inside the sphere.[3]

**13a22 Exercise.** The density of a "material" sphere of radius $R$ is proportional to the distance to the vertical diameter. Find the centroid of the upper hemisphere.[4]

**13a23 Exercise.** Find the centroid of the (homogeneous) conic surface $0 < z = \sqrt{x^2 + y^2} < 1$.[5]

Vector-valued functions may be integrated as well. Given $f : M \to \mathbb{R}^\ell$, $f : x \mapsto \big(f_1(x), \dots, f_\ell(x)\big)$, we define

$$\int_M f = \bigg( \int_M f_1, \dots, \int_M f_\ell \bigg) \tag{13a24}$$

provided that these $\ell$ integrals are well-defined. Accordingly, for $f : M \to V$ where $V$ is an $\ell$-dimensional vector space, we define $\int_M f$ by

$$L\bigg( \int_M f \bigg) = \int_M L \circ f \quad \text{for all linear } L : V \to \mathbb{R} \,, \tag{13a25}$$

provided that the right-hand side is well-defined for all $L$.

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[1] Hint: use 13a15.

[2] Answer: 8.

[3] Answer: (a) $2\pi - 4$; (b) 8.

[4] Answer: $(0, 0, \frac{4}{3\pi} R)$.

[5] Answer: $(0, 0, \frac{2}{3})$.

## 13b Integral of derivative

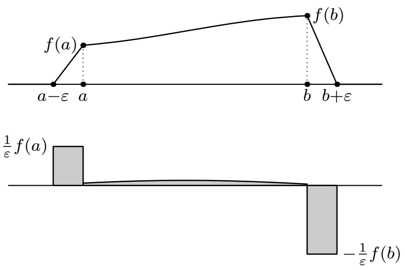
Everyone knows that

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)\,. \tag{13b1}$$

And in particular,

$$\int_{\mathbb{R}} f'(x)\,\mathrm{d}x = 0 \quad \text{if } f \in C^1(\mathbb{R}) \text{ has a bounded support.} \tag{13b2}$$

Interestingly, (13b1) may be thought of as (13b2) applied to the function $f \cdot 1\!\!1_{[a,b]}$. Yes, this function is not differentiable (and moreover, not continuous), but let us approximate it:

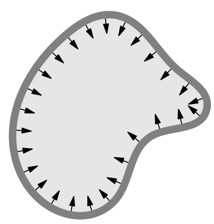



$$f(a) + \int_a^b f'(x)\,\mathrm{d}x - f(b) = 0\,.$$

This simple idea becomes much more interesting in higher dimensions. The equality (13b1) has no evident $n$-dimensional counterpart, but (13b2) has:

$$\int_{\mathbb{R}^n} \nabla f = 0 \quad \text{if } f \in C^1(\mathbb{R}^n) \text{ has a bounded support.} \tag{13b3}$$

Given a geometric body $E \subset \mathbb{R}^n$, we approximate its indicator $1\!\!1_E$ and take the gradient. In the boundary layer of thickness $\varepsilon$ we get the inwards normal vector of length $1/\varepsilon$. In the limit (for $\varepsilon \to 0+$) we see that the integral over $\partial E$ of the (inwards) normal unit vector must vanish. This is indeed true and useful. However, the limiting procedure, helpful for intuition, is less helpful for the proof (which happens often in mathematics).



Now we finish this informal prelude and start the formal theory.

We take the case $n = N - 1$; that is, we consider an $n$-dimensional manifold $M$ in $\mathbb{R}^{n+1}$, often called a *hypersurface*. In this case, two sides of $M$ will be defined locally (but not globally; think about the Möbius strip), after some preparation.

Given $x_0 \in M$, we consider germs[1] $[\sigma]$ (at $x_0$) of functions $\sigma : \mathbb{R}^N \setminus M \to \mathbb{R}$ that are continuous near $x_0$ and satisfy $\sigma(\cdot) = \pm 1$ near $x_0$.

**13b4 Lemma.** There exist exactly 4 such germs; two are constant; the other two are not, and these two are mutually opposite ($[\sigma]$ and $[-\sigma]$).

***Proof.*** The conditions on $\sigma$ (and $M$) are invariant under local homeomorphisms. By 12b4(c), WLOG we assume that $M$ is the hyperplane $\mathbb{R}^n \times \{0\}$. We take $\sigma$ of the germ and $\varepsilon > 0$ such that $\sigma$ is continuous and equal $\pm 1$ on the set $\{x \in \mathbb{R}^N \setminus M : |x - x_0| < \varepsilon\}$, and note that this set has exactly two connected components. □

From now on, $[\sigma]$ stands for one of the two non-constant germs; let us call it *side indicator.*

**13b5 Definition.** A function $f : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}$ is *continuous up to* $M$, if it is continuous (on $\mathbb{R}^N \setminus \overline{M}$) and for every $x_0 \in M$ the limits

$$f_-(x) = \lim_{y \to x, \sigma(y) = -1} f(y) \quad \text{and} \quad f_+(x) = \lim_{y \to x, \sigma(y) = +1} f(y)$$

exist for all $x \in M$ near $x_0$.

In this case the germs $[f_-], [f_+]$ (of functions on $M$) are well-defined and continuous. The difference $f_+(x_0) - f_-(x_0)$ of these "one-sided limits" at $x_0$ is the *jump* of $f$ at $x_0$. Its sign depends on the sign of $\sigma$.

The same applies when $M$ is an $n$-dimensional manifold in an $(n+1)$-dimensional affine space. In contrast, the unit normal vector and the singular gradient, defined below, require Euclidean metric.

The tangent space $T_x M$, being a hyperplane in $\mathbb{R}^N$, is

$$T_x M = \{h : \langle h, \mathbf{n}_x \rangle = 0\}$$

for some unit vector $\mathbf{n}_x \in \mathbb{R}^N$, the so-called *unit normal vector.* It is well-defined up to the sign. When using together $\mathbf{n}_x$ and the side indicator we always assume that they conform:

$$\sigma(x + \lambda \mathbf{n}_x) = \begin{cases} -1 & \text{for small } \lambda < 0, \\ +1 & \text{for small } \lambda > 0. \end{cases}$$

Thus we have a germ of a mapping $x \mapsto \mathbf{n}_x$. It is continuous due to an explicit formula given in the following exercise. (However, a continuous mapping $x \mapsto \mathbf{n}_x$ on the whole $M$ exists if and only if $M$ is orientable; we'll return to this point much later.)

[1] Recall Sect. 1c.

**13b6 Exercise.** Let $M$ be locally the graph,

$$\{(x_1,\dots,x_N) : x_N = g(x_1,\dots,x_n)\}\,,$$

of a continuously differentiable function $g$. Then the formula

$$\sigma(x_1,\dots,x_N) = \begin{cases} -1 & \text{for } x_N < g(x_1,\dots,x_n), \\ +1 & \text{for } x_N > g(x_1,\dots,x_n) \end{cases}$$

defines a side indicator, and the formula

$$\mathbf{n}_x = \frac{1}{\sqrt{1+|\nabla g|^2}}\big(-(D_1 g),\dots,-(D_n g),1\big)$$

defines the corresponding unit normal vector.

Prove it.[1]

Here is a more convenient notation for the one-sided limits:

$$f(x-0\mathbf{n}_x) = f_-(x) \quad \text{and} \quad f(x+0\mathbf{n}_x) = f_+(x)\,.$$

**13b7 Definition.** The *singular gradient*[2] $\nabla_{\text{sng}} f(x)$ at $x \in M$ of a function $f : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}$ continuous up to $M$ is the vector

$$\nabla_{\text{sng}} f(x) = \big(f(x+0\mathbf{n}_x) - f(x-0\mathbf{n}_x)\big)\mathbf{n}_x\,.$$

Note that the singular gradient does not depend on the sign of $\sigma$ (and $\mathbf{n}_x$). It is a continuous mapping $\nabla_{\text{sng}} f : M \to \mathbb{R}^N$. (Think, what happens if $M$ is the Möbius strip.)

A compact subset of an $n$-manifold in $\mathbb{R}^N$ is of ($N$-dimensional) volume zero[3] (recall 13a10(a)). However, this may fail for a bounded subset. When a manifold $M$ is not a closed set,[4] it may be rather wild near a point of $\overline{M} \setminus M$.

**13b8 Example.** A bounded 1-manifold in $\mathbb{R}^2$ need not be a set of area zero.

Similarly to Example 8b9, we start with a sequence of pairwise disjoint closed intervals $[s_1,t_1],[s_2,t_2],\dots \subset (0,1)$ such that $\sum_k (t_k - s_k) = a < 1$ and the open set $G = (s_1,t_1) \cup (s_2,t_2) \cup \dots$ is dense in $(0,1)$.[5] The set $M_0 = \{\frac{s_k+t_k}{2} : k = 1,2,\dots\}$ of the centers of these intervals is a 0-manifold

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[1] Hint: $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=0}\varphi(x+\lambda\mathbf{n}_x)$ where $\varphi(x_1,\dots,x_N) = x_N - g(x_1,\dots,x_n)$.

[2] Not a standard terminology.

[3] Except for the case $n = N$, of course.

[4] Be warned: "The notion of closed manifold is unrelated with that of a closed set." Wikipedia:Closed manifold#Contrasting terms

[5] Its complement $[0,1] \setminus G$ is sometimes called a fat Cantor set.

in $\mathbb{R}$ (a discrete set). Its closure contains $[0,1] \setminus G$; thus, $v^*(M_0) = v^*(\overline{M}_0) = 1 - a > 0$.

The set $M_1 = M_0 \times (0,1)$ is a 1-manifold in $\mathbb{R}^2$ (recall 12b9), not of area zero.

**13b9 Theorem.** Let $M \subset \mathbb{R}^{n+1}$ be an $n$-manifold, $K \subset M$ a compact subset, and $f : \mathbb{R}^{n+1} \setminus K \to \mathbb{R}$ a function such that

(a) $f$ is continuously differentiable (on $\mathbb{R}^{n+1} \setminus K$);

(b) $f|_{\mathbb{R}^{n+1} \setminus \overline{M}}$ is continuous up to $M$;

(c) $f$ has a bounded support, and $\nabla f$ is bounded (on $\mathbb{R}^{n+1} \setminus K$).

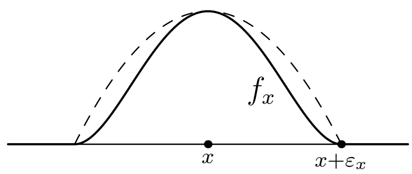
Then

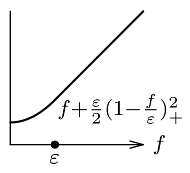
$$\int_{\mathbb{R}^{n+1} \setminus K} \nabla f + \int_M \nabla_{\text{sng}} f = 0 \,.$$

**13b10 Remark.** First, both integrands being vector-valued, both integrals are treated as in (13a24)–(13a25). Second, $K$ is of volume zero, and (c) implies integrability of $\nabla f \cdot 1\!\!1_{\mathbb{R}^{n+1} \setminus K}$ (think, why).[1] Third, $\nabla_{\text{sng}} f$ is continuous and compactly supported (by $K$) on $M$ (think, why).

**13b11 Lemma.** Let $(U_1, \dots, U_\ell)$ be an open covering of a compact set $K \subset \mathbb{R}^N$. Then there exist functions $\rho_1, \dots, \rho_i \in C^1(\mathbb{R}^N)$ such that $\rho_1 + \dots + \rho_i = 1$ on $K$ and each $\rho_j$ has a compact support within some $U_m$.

***Proof.*** Similar to the proof of Lemma 13a3, with the following modifications. First, the sets $U_1, \dots, U_\ell$ are used instead of charts. Second, functions $f_x : y \mapsto (\varepsilon_x^2 - |y - x|^2)_+^2$ are used instead of $y \mapsto (\varepsilon_x - |y - x|)_+$, in order to ensure continuous differentiability.




Third (for the same reason), $f(x) + \frac{\varepsilon}{2}\big(1 - \frac{f(x)}{\varepsilon}\big)_+^2$ is used in the denominator of $\rho_k$ instead of $\max(f(x), \varepsilon)$. □

Still, $N = n + 1$, and $M, K, f$ are as in Theorem 13b9.

**13b12 Lemma.** Let $h \in \mathbb{R}^N$ be such that

$$\forall x \in K \;\; h \notin T_x M \,.$$

[1] Lebesgue's criterion 8f1 applies. But really, a much simpler argument suffices, similar to that of 6f2, since "volume zero" is much stronger than "Lebesgue measure zero".

Then

$$\Big\langle h, \int_{\mathbb{R}^N \setminus K} \nabla f + \int_M \nabla_{\text{sng}} f \Big\rangle = 0 \,.$$

***Proof.*** WLOG, $h = (0, \dots, 0, 1)$ is the last vector of the usual basis of $\mathbb{R}^N$ (otherwise, downgrade $\mathbb{R}^N$ to a Euclidean vector space, and upgrade back to $\mathbb{R}^N$ as needed).

For every $x \in K$ we take a co-chart $(U, \varphi)$ of $M$ around $x$. By 12b19(c), $(D\varphi)_x h \ne 0$, that is, $(D_N \varphi)_x \ne 0$. The implicit function theorem 5c1 gives us an open set $U \subset \mathbb{R}^n$ and an open interval $V \subset \mathbb{R}$ such that $x \in U \times V$ and $M \cap (U \times V)$ is the graph of a function $U \to V$ (of class $C^1$). Such $U \times V$ are an open covering of $K$. We take a finite subcovering $W_1, \dots, W_\ell$, add one more open set $W_0 = \mathbb{R}^N \setminus K$, and get a finite open covering of a closed ball $B$ that supports $f$. Lemma 13b11 gives $\rho_1, \dots, \rho_i \in C^1(\mathbb{R}^N)$ such that $\rho_1 + \dots + \rho_i = 1$ on $B$ and each $\rho_j$ has a compact support within some $W_m$. Taking into account linearity of integrals and gradients we reduce the claim for $f$ to the same claim for $\rho_1 f, \dots, \rho_i f$. Thus, WLOG, we may assume that $f$ has a compact support either within $\mathbb{R}^N \setminus K$ or within some $U \times V$.

If $f$ has a compact support within $\mathbb{R}^N \setminus K$ then we extend $f$ to $K$ as 0 and get $f \in C^1(\mathbb{R}^N)$, $\nabla_{\text{sng}} f = 0$, $\int_{\mathbb{R}^N \setminus K} \nabla f = \int_{\mathbb{R}^N} \nabla f = 0$ by (13b3).

It remains to consider $f$ that has a compact support within some $U \times V$, $V = (a, b)$, such that $M \cap (U \times V)$ is the graph of some $g \in C^1(U)$, $g : U \to (a, b)$. On one hand, taking into account that $\nabla f$ is integrable,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus K} \langle h, \nabla f \rangle &= \int_{U \times (a,b) \setminus K} D_N f = \\ &= \int_U \mathrm{d}u_1 \dots \mathrm{d}u_n \bigg( \int_a^{g(u)} + \int_{g(u)}^b \bigg) \mathrm{d}t \frac{\partial}{\partial t} f(u_1, \dots, u_n, t) = \\ &= \int_U \big( f(u, g(u)-) - f(u, g(u)+) \big) \, \mathrm{d}u \,. \end{aligned}$$

On the other hand, using the side indicator

$$\sigma(u,t) = \begin{cases} -1 & \text{for } t < g(u), \\ +1 & \text{for } t > g(u) \end{cases} \quad \text{for } u \in U \text{ and } t \in (a, b) \,,$$

we have for $u \in U$ and $x = (u, g(u))$

$$\mathbf{n}_x = \frac{1}{\sqrt{1 + |\nabla g(u)|^2}} \big( -(D_1 g)_u, \dots, -(D_n g)_u, 1 \big)$$

(recall 13b6); thus,

$$\langle h, \mathbf{n}_x \rangle = \frac{1}{\sqrt{1+|\nabla g(u)|^2}}\,;$$
$$f(x-0\mathbf{n}_x) = f\big(u, g(u)-\big)\,, \quad f(x+0\mathbf{n}_x) = f\big(u, g(u)+\big)\,;$$
$$\langle h, \nabla_{\text{sng}} f(x) \rangle = \frac{f\big(u,g(u)+\big) - f\big(u,g(u)-\big)}{\sqrt{1+|\nabla g(u)|^2}}\,;$$

and finally, using 12c19,

$$\int_M \langle h, \nabla_{\text{sng}} f \rangle = \int_U \frac{f\big(u,g(u)+\big) - f\big(u,g(u)-\big)}{\sqrt{1+|\nabla g(u)|^2}} \sqrt{1+|\nabla g(u)|^2} = \\ = \int_U \big( f(u,g(u)+) - f(u,g(u)-) \big)\, \mathrm{d}u\,.$$

□

***Proof** of Theorem 13b9.* Every point $x_0 \in M$ has a neighborhood $U$ such that[1]

$$|\langle \mathbf{n}_x, \mathbf{n}_y \rangle| \ge \frac12 \quad \text{for all } x, y \in M \cap U$$

(since $y \mapsto \mathbf{n}_y$ is continuous near $x$). A partition of unity (used similarly to the proof of 13b12) reduces the claim for $f$ to the same claim for $\rho f$ where $\rho \in C^1(\mathbb{R}^N)$ has a compact support either within $\mathbb{R}^N \setminus K$ or within some $U$. The former case is trivial (as before); consider the latter case: $\rho$ has a compact support within $U$. We introduce $\tilde K = K \cap \overline{U}$, extend $\rho f$ to $\mathbb{R}^N \setminus \tilde K$ as 0 on $K \setminus \tilde K$, and observe that $\tilde K$ and $\rho f$ satisfy the conditions of Theorem 13b9. Thus (renaming $\tilde K$ and $\rho f$ into $K$ and $f$), WLOG,

$$|\langle \mathbf{n}_x, \mathbf{n}_y \rangle| \ge \frac12 \quad \text{for all } x, y \in K\,.$$

We choose $x_0 \in K$ and note that every $h \in \mathbb{R}^N$ such that $|h - \mathbf{n}_{x_0}| < 1/2$ satisfies the condition of Lemma 13b12, since

$$h \in T_x M \implies \langle h, \mathbf{n}_x \rangle = 0 \implies |\langle \mathbf{n}_{x_0}, \mathbf{n}_x \rangle| = |\langle \mathbf{n}_{x_0} - h, \mathbf{n}_x \rangle| < \frac12 \implies x \notin K\,.$$

By 13b12, the vector $\int_{\mathbb{R}^N \setminus M} \nabla f + \int_M \nabla_{\text{sng}} f$ is orthogonal to all these $h$, and therefore, equal to zero. □

[1] Any number of $(0,1)$ may be used instead of $1/2$.

Often $f = uv$ where $u, v$ both satisfy the same conditions (a,b,c) of Theorem 13b9. Then $\nabla f = u\nabla v + v\nabla u$ (by the product rule 2b8), thus,

$$\int_{\mathbb{R}^N\setminus K} u\nabla v = -\int_{\mathbb{R}^N\setminus K} v\nabla u - \int_M \nabla_{\text{sng}}(uv)\,; \tag{13b13}$$

this is a kind of high-dimensional integration by parts. And, of course,

$$\int_{\mathbb{R}^N} u\nabla v = -\int_{\mathbb{R}^N} v\nabla u \tag{13b14}$$

for $u, v \in C^1(\mathbb{R}^N)$ such that $uv$ is compactly supported.

Often, a hypersurface $M$ is a boundary $\partial G = \overline{G} \setminus G$ of an open set $G \subset \mathbb{R}^N$. It may seem that in this case $M$ must be orientable, with two sides (globally), inner and outer; but this is generally wrong. A manifold $M$ (not just a hypersurface) is a boundary of some $G$ if and only if $M$ is a closed set. Here is why. On one hand, $\partial G$ is always closed. On the other hand, given a closed $M$, we may take $G = \mathbb{R}^N \setminus M$ and get $\partial G = M$ (even for the non-orientable compact $M$ of 12b13). Boundedness of $G$ does not help; if $G = B \setminus M$ where $B \supset M$ is an open ball, then $\partial G$ consists of $M$ and the sphere $\partial B$.

In fact, if a hypersurface is a closed set, then it is orientable;[1] but even in this case both sides may be inner for a given $G$ (try $G = \mathbb{R}^N \setminus M$ or $G = B \setminus M$ again).

An open set $G \subset \mathbb{R}^N$ is called *regular,* if $(\overline{G})^\circ = G$; that is, the interior of the closure of $G$ is equal to $G$. (Generally it cannot be less than $G$, but can be more than $G$; a simple example: $G = \mathbb{R} \setminus \{0\}$.) Equivalently, $G$ is regular if (and only if) $\partial G = \partial(\mathbb{R}^N \setminus \overline{G})$; that is, the boundary of the exterior of $G$ is equal to the boundary of $G$.

From now on (till the end of 13b), $G \subset \mathbb{R}^N$ is a bounded regular open set, and $\partial G = M \subset \mathbb{R}^N$ a (necessarily compact) hypersurface (that is, $n$-manifold for $n = N - 1$).

It follows that $\partial G$ is of volume zero; by 8d4(b), $G$ is Jordan measurable. The function

$$\sigma(x) = \begin{cases} -1 & \text{for } x \in G, \\ +1 & \text{for } x \notin \overline{G} \end{cases}$$

is a side indicator on the whole $M$. The corresponding *outward* unit normal vector $\mathbf{n}_x$ satisfies

$$\begin{aligned} x + \lambda\mathbf{n}_x &\in G \quad \text{for small } \lambda < 0\,, \\ x + \lambda\mathbf{n}_x &\notin G \quad \text{for small } \lambda > 0\,. \end{aligned}$$

[1] See for instance "Orientability of hypersurfaces in $\mathbb{R}^n$" by H. Samelson, Proc. Amer. Math. Soc. **22**:1 (1969) 301–302.

Let $f : \overline{G} \to \mathbb{R}$ be continuous, $f|_G \in C^1(G)$, with $\nabla f$ bounded (on $G$). Then the function $\tilde{f} : \mathbb{R}^N \setminus M \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin \overline{G} \end{cases}$$

is continuous up to $M$, and

$$\tilde{f}(x - 0\mathbf{n}_x) = f(x), \quad \tilde{f}(x + 0\mathbf{n}_x) = 0;$$
$$\nabla_{\text{sng}} \tilde{f}(x) = -f(x)\mathbf{n}_x.$$

By Theorem 13b9 (applied to $\tilde{f}$ and $K = M$),

$$\int_G \nabla f = \int_M f\mathbf{n}. \tag{13b15}$$

In particular, for $f(\cdot) = 1$,

$$\int_M \mathbf{n} = 0; \tag{13b16}$$

and for a linear function $f : x \mapsto \langle h, x \rangle$,

$$\int_M \langle h, \cdot \rangle \mathbf{n} = v(G)h \quad \text{for } h \in \mathbb{R}^N. \tag{13b17}$$

Interestingly, (13b17) for $N = 3$ is basically Archimedes' principle: the upward buoyant force that is exerted on a body immersed in a fluid, is equal to the weight of the fluid that the body displaces.[1] Here is why. At a point $(x, y, z) \in \mathbb{R}^2 \times (-\infty, 0) \subset \mathbb{R}^3$, the depth below the surface of water being $(-z)$, the hydrostatic pressure is $\rho g(-z)$, where $\rho$ is the water density, and $g \approx 9.8\,\mathrm{m/s}^2$ is the gravitational acceleration. Infinitesimally, the force per unit area is $\rho g(-z)(-\mathbf{n}_{(x,y,z)}) = \langle h, (x, y, z) \rangle \mathbf{n}_{(x,y,z)}$ where $h = \rho g(0, 0, 1)$. The total force is $\int_M \langle h, \cdot \rangle \mathbf{n} = v(G)h = \rho g v(G)(0, 0, 1)$, the weight of the mass $\rho v(G)$ of the displaced water, directed upwards.

## 13c Curvilinear iterated integral

Recall several facts.

∗ The iterated integral approach (Sect. 7) decomposes an integral over the plane into integrals over parallel lines. It also decomposes an integral over 3-dimensional space into integrals over parallel planes.[2]

[1] Wikipedia:Archimedes' principle.

[2] Or alternatively, parallel lines. In this course we restrict ourselves to dimension $n + 1$; for dimension $n + m$ see the "Coarea formula" in Encyclopedia of Math.

* The idea of a curvilinear iterated integral was discussed a bit in Remark 7d13.
* A 3-dimensional integral decomposes into integrals over spheres, see 12c26(b).
* However, a naive attempt to decompose an integral over the plane into integrals over curves $y = f(x)+a$ fails (see 12c22); a new factor appears, like Jacobian.

Thus, we want to understand, whether or not a 2-dimensional integral decomposes into integrals over curves $\varphi(\cdot) = \text{const}$, and what about a new factor; and what happens in dimension 3 (and more).

**13c1 Theorem.** Let $G \subset \mathbb{R}^{n+1}$ be an open set, $\varphi \in C^1(G)$, $\forall x \in G\ \nabla\varphi(x) \ne 0$, and $f \in C(G)$ compactly supported. Then for every $c \in \varphi(G)$ the set $M_c = \{x \in G : \varphi(x) = c\}$ is an $n$-manifold in $\mathbb{R}^{n+1}$, the function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous and compactly supported, and

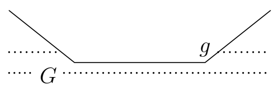
$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} f = \int_G f|\nabla\varphi| \,.$$

**13c2 Remark.** The open sets $G \subset \mathbb{R}^{n+1}$ and $\varphi(G) \subset \mathbb{R}$ need not be Jordan measurable, but still, the integrals are well-defined, since $f$ is supported by some compact $K \subset G$, and the function $c \mapsto \int_{M_c} f$ is supported by the compact $\varphi(K) \subset \varphi(G)$.

The new factor $|\nabla\varphi|$ shows roughly, how many hypersurfaces $M_c$ intersect an infinitesimal neighborhood of a point.

The set $M_c$ is an $n$-manifold, just because $(G, \varphi(\cdot) - c)$ is a co-chart of the whole $M_c$. But continuity of the function $c \mapsto \int_{M_c} f$ is a harder matter.

**13c3 Remark.** A function $c \mapsto v(M_c)$ need not be continuous on $\varphi(G)$. For a counterexample try $G = \{(x, y) : y < g(x)\} \subset \mathbb{R}^2$ and $\varphi(x, y) = y$.




**13c4 Lemma.** The function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous.

***Proof.*** If $x \in G$ satisfies $(D_N\varphi)_x \ne 0$, then the mapping $h : (\tilde{x}_1, \dots, \tilde{x}_N) \mapsto \big(\tilde{x}_1, \dots, \tilde{x}_n, \varphi(\tilde{x}_1, \dots, \tilde{x}_N)\big)$ is a local diffeomorphism near $x$ (recall the proof of the implicit function theorem), and we may take open neighborhoods $U$ of $x$, $V$ of $(x_1, \dots, x_n)$ and $W$ of $\varphi(x)$ such that $h$ is a diffeomorphism between $U$ and $V \times W$.

If $(D_N\varphi)_x = 0$, we just use another coordinate in the same way.

Using a partition of unity (similarly to the proof of 13b12) we see that, WLOG, $f$ is supported by $U$ such that $h : U \to V \times W$ is a diffeomorphism.

Now, $M_c \cap U$ is the graph of the function $g_c$ defined by $h^{-1}(x_1,\dots,x_n,c) = \big(x_1,\dots,x_n,g_c(x_1,\dots,x_n)\big)$. Using 12c19,

$$\int_{M_c} f = \int_V f\big(x_1,\dots,x_n,g_c(x_1,\dots,x_n)\big)\sqrt{1+|\nabla g_c(x_1,\dots,x_n)|^2}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

The integrand is uniformly continuous on the relevant compact set, therefore the integral is continuous. □

**13c5 Lemma.**

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} \frac{f}{|\nabla\varphi|}\nabla\varphi = \int_G f\nabla\varphi$$

for $G, \varphi, f$ and $M_c$ as in Theorem 13c1.

***Proof.*** Let $K \subset G$ be a compact set that supports $f$. Clearly, $\varphi$ is bounded on $K$. WLOG, there exists $C > 0$ such that $0 < \varphi(\cdot) < C$ on $K$ (since we may add a large constant to $\varphi$). WLOG, $f \in C^1(G)$, since it can be approximated uniformly by functions of class $C^1$ supported by a small neighborhood of $K$ (and the volume of the relevant part of $M_c$ is bounded in $c$).

Given $c \in (0,C) \cap \varphi(G)$, we introduce $G_c = \{x \in G : \varphi(x) > c\}$, $K_c = K \cap M_c$ (empty, if $c \notin \varphi(K)$), and define $f_c : \mathbb{R}^N \setminus K_c \to \mathbb{R}$ by

$$f_c(x) = \begin{cases} f(x) & \text{if } x \in G_c, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}^N \setminus K_c$. Clearly, $f_c$ is continuously differentiable (on $\mathbb{R}^N \setminus K_c$), with bounded gradient, and

$$\nabla f_c(x) = \begin{cases} \nabla f(x) & \text{if } x \in G_c, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}^N \setminus K_c$.

Using the side indicator

$$\sigma(x) = \begin{cases} -1 & \text{if } \varphi(x) < c, \\ +1 & \text{if } \varphi(x) > c \end{cases}$$

and the unit normal vector

$$\mathbf{n}_x = \frac{1}{|\nabla\varphi(x)|}\nabla\varphi(x)\,,$$

we see that $f_c$ is continuous up to $M_c$,

$$f_c(x - 0\mathbf{n}_x) = 0\,, \quad f_c(x + 0\mathbf{n}_x) = f(x)\,;$$
$$\nabla_{\rm sng} f_c(x) = f(x)\mathbf{n}_x = \frac{f(x)}{|\nabla\varphi(x)|}\nabla\varphi(x)\,.$$

By Theorem 13b9,

$$\int_{\mathbb{R}^N\setminus K_c} \nabla f_c + \int_{M_c} \nabla_{\rm sng} f_c = 0\,,$$

that is,

$$\int_{G_c} \nabla f + \int_{M_c} \frac{f}{|\nabla\varphi|}\nabla\varphi = 0\,.$$

Now we have to integrate it in $c$. We apply the iterated integral to the function $G \times (0,C) \to \mathbb{R}^N$, $(x,c) \mapsto 1\!\!1_{G_c}(x)\nabla f(x)$, integrable since it is discontinuous only on the set $\{(x,\varphi(x)) : x \in K\}$ of volume zero; we get

$$\int_0^C \mathrm{d}c \int_{G_c} \mathrm{d}x\, \nabla f(x) = \int_G \mathrm{d}x\, \nabla f(x) \underbrace{\int_0^C 1\!\!1_{G_c}(x)}_{\varphi(x)} = \int_G \varphi\nabla f\,.$$

By (13b14), $\int_G \varphi\nabla f = -\int_G f\nabla\varphi$. It remains to note that $\int_{G_c}\nabla f = 0$ for $c \in (0,C)\setminus\varphi(G)$, since in this case $f_c \in C^1(G)$. □

***Proof** of Theorem 13c1.* Using a partition of unity (similarly to the proof of 13b12) we see that, WLOG, there exists $h \in \mathbb{R}^N$ such that $|h| = 1$ and $D_h\varphi > 0$ on a compact $K \subset G$ that supports $f$. Applying Lemma 13c5 to the function $\frac{f|\nabla\varphi|}{\langle h,\nabla\varphi\rangle}$ we get

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} \frac{f|\nabla\varphi|}{\langle h,\nabla\varphi\rangle}\frac{\nabla\varphi}{|\nabla\varphi|} = \int_G \frac{f|\nabla\varphi|}{\langle h,\nabla\varphi\rangle}\nabla\varphi\,.$$

It remains to take the scalar product by $h$. □

**13c6 Exercise.** Apply Theorem 13c1 to $G = \mathbb{R}^2$, $\varphi(x,y) = y - \sin x$; compare the result with 7b9. Do they agree?

**13c7 Exercise.** (a) Apply Theorem 13c1 to $G = \mathbb{R}^2 \setminus \{0\}$, $\varphi(x) = |x|$; compare the result with 9b2 (polar coordinates). Do they agree?

(b) The same for spherical coordinates (recall 9b3).

**13c8 Exercise.** (a) $\displaystyle\int_0^\infty \mathrm{d}r \int_{|\cdot|=r} f = \int_{|\cdot|>0} f$
for all compactly supported $f \in C(\mathbb{R}^N \setminus \{0\})$.

(b) Generalize it (formulate accurately, and prove) for all integrable $f$ on $\mathbb{R}^N$.[1]

Taking $f(x) = 1$ for $|x| < 1$, otherwise 0, and using (10d7), we get $\int_0^1 v(S_r)\,\mathrm{d}r = v(B_1) = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ where $S_r = \{x : |x| = r\}$ is a sphere, and $B_1 = \{x : |x| < 1\}$ a ball. By 13a14, $v(S_r) = r^{N-1}v(S_1)$. Thus, $v(B_1) = \int_0^1 r^{N-1}v(S_1)\,\mathrm{d}r = \frac{1}{N}v(S_1)$;

$$v(S_1) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,. \tag{13c9}$$

**13c10 Exercise.** Find the $(N-1)$-dimensional volume of the simplex $M = \{x \in (0,\infty)^N : x_1 + \dots + x_N = 1\}$ in $\mathbb{R}^N$.[2]

**13c11 Exercise.** Integrate the function $x \mapsto x_1^{p_1} \dots x_N^{p_N}$ over the hypersurface $S_+ = \{x \in (0,\infty)^N : |x| = 1\}$ (the positive part of the sphere) in $\mathbb{R}^N$ for $p_1, \dots, p_N \in (-1, \infty)$.[3]

**13c12 Exercise.** Find $\displaystyle\int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(1+|x|^2)^p}$ for $p \in \left(\frac{n}{2}, \infty\right)$.[4]

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[1] Hint: (a) $\varphi(x) = |x|$; (b) similar to Theorem 7d1.

[2] Answer: $\sqrt{n}/(n-1)!$. Hint: similar to (13c9); use 10g1 for $p_1 = \dots = p_n = 1$.

[3] Answer: $\frac{\Gamma(\frac{p_1+1}{2})\dots\Gamma(\frac{p_N+1}{2})}{2^{N-1}\Gamma(\frac{p_1+\dots+p_N+N}{2})}$. Hint: $\int_{(0,\infty)^N} \mathrm{e}^{-|x|^2} x_1^{p_1} \dots x_N^{p_N}\,\mathrm{d}x$.

[4] Answer: $\pi^{n/2}\frac{\Gamma(p-\frac{n}{2})}{\Gamma(p)}$. Hint: use (10d8). Really, you can do it without manifolds, in the spirit of 10b7.

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