6 Basics of integration

Integral is a bridge between functions of point and functions of set.

6a What is the problem

A quote:

> As already pointed out, many of the quantities of interest in continuum mechanics represent *extensive properties,* such as mass, momentum and energy. An extensive property assigns a value to *each part of the body.* From the mathematical point of view, an extensive property can be regarded as a *set function,* in the sense that it assigns a value to each subset of a given set. Consider, for example, the case of the mass property. Given a material body, this property assigns to each subbody its mass. Other examples of extensive properties are: volume, electric charge, internal energy, linear momentum. *Intensive properties,* on the other hand, are represented by *fields,* assigning to *each point of the body* a definite value. Examples of intensive properties are: temperature, displacement, strain.
>
> As the example of mass clearly shows, very often the extensive properties of interest are *additive set functions,* namely, the value assigned to the union of two disjoint subsets is equal to the sum of the values assigned to each subset separately. Under suitable assumptions of continuity, it can be shown that an additive set function is expressible as the integral of a *density* function over the subset of interest. This density, measured in terms of property per unit size, is an ordinary pointwise function defined over

> the original set. In other words, the density associated with a continuous additive set function is an intensive property. Thus, for example, the mass density is a scalar field.
>
> Marcelo Epstein[1]

We need a mathematical theory of the correspondence between set functions $\mathbb{R}^n \supset E \mapsto F(E) \in \mathbb{R}$ and (ordinary) functions $\mathbb{R}^n \ni x \mapsto f(x) \in \mathbb{R}$ via integration, $F(E) = \int_E f$. The theory should address (in particular) the following questions.

* What are admissible sets E and functions f? (Arbitrary sets are as useless here as arbitrary functions.)
* What is meant by "disjoint"?
* What is meant by integral?
* What are the general properties of the integral?
* How to calculate the integral explicitly for given f and E ?

We start the integration theory based on two postulates. First,

(6a1) $$\operatorname{vol}(B) \inf_B f \le F(B) \le \operatorname{vol}(B) \sup_B f$$

whenever B is a box (to be defined). Second,

(6a2) $$F(B_1 \cup \dots \cup B_k) = F(B_1) + \dots + F(B_k)$$

whenever a box B is split into k boxes $B_1, \dots, B_k$.

For boxes the theory is similar to the one-dimensional Riemann integration. However, two problems need additional effort:

* E need not be a box (it may be a ball, a cone, etc.);
* rotation invariance should be proved.

These problems do not appear in dimension one; there an (ordinary) function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$ leads to the set function $[s, t] \mapsto F(t) - F(s) = \int_s^t f$.

6b Dimension one, revisited

> It is frequently claimed that Lebesgue integration is as easy to teach as Riemann integration. This is probably true, but I have yet to be convinced that it is as easy to learn.
>
> T.W. Körner[2]

[1] "The elements of continuum biomechanics", Wiley 2012. (See Sect. 2.2.1.)

[2] "A companion to analysis: A second first and first second course in analysis", AMS 2004. (See page 197.) Among our textbooks Shurman, Shirfin and Zorich treat Riemann integral, Fleming treats Lebesgue integral, and Hubbard treats both.

One-dimensional Riemann integration was treated in Analysis-II; the integral $\int_s^t f(x)\,\mathrm{d}x$, defined for *integrable* functions $f:[s,t]\to\mathbb{R}$, is additive in the interval $[s,t]$ and linear in the function f. Now we'll do a bit more: the lower and upper integrals $_*\!\int_{[s,t]} f$, $^*\!\!\int_{[s,t]} f$ will be defined for all *bounded* functions $f:[s,t]\to\mathbb{R}$; they are additive in the interval $[s,t]$ but not linear in the function f; and they are equal if and only if f is integrable.

We treat dimension 1 as a special case of dimension n (treated later); this is why our terminology and notation are rather ugly in dimension 1.

Intervals $[s,t]$, (s,t), $[s,t)$ and $(s,t]$ for $-\infty<s<t<\infty$ will be called 1-dimensional *boxes* and denoted by B (or C). We do not care whether B is open, closed, or neither; instead, we use the closure $\overline{B}$ or the interior B° as needed. The length $t-s$ of the box will be called its 1-dimensional *volume* and denoted by $\operatorname{vol}(B)$.

A finite subset of B° divides B into finitely many subintervals; the set P of these subintervals will be called a *partition* of B;

(6b1) $$\overline{B}=\bigcup_{C\in P}\overline{C}\,;\quad C_1^\circ\cap C_2^\circ=\emptyset \text{ for } C_1,C_2\in P,\ C_1\ne C_2\,.$$

The volume is *additive:*

(6b2) $$\operatorname{vol}(B)=\sum_{C\in P}\operatorname{vol}(C)\,.$$

Adding more points to the finite subset of B° we get a *refinement* P' of the partition P; this is another partition such that for every $C\in P$,

(6b3) $$\text{the set}\quad P'|_C=\{C'\in P':C'\subset C\}\quad\text{is a partition of } C\,,$$

and these fragments together give P':

(6b4) $$P'=\biguplus_{C\in P}P'|_C\,.$$

In particular, P is a refinement of itself.

The union of two finite subsets of B° leads to the *common refinement* P_1P_2 of two partitions P_1 and P_2:

(6b5) $$P_1P_2=\{C_1\cap C_2:C_1\in P_1,C_2\in P_2,C_1^\circ\cap C_2^\circ\ne\emptyset\}\,.$$

Thus, P' is a refinement of P if and only if $P'=PP'$.

Now let a function $f:\mathbb{R}\to\mathbb{R}$ be locally bounded, that is, bounded on every box (or, equivalently, on every bounded subset of $\mathbb{R}$). For arbitrary box $B\subset\mathbb{R}$ we consider

$$\sup_{\overline{B}}f=\sup_{x\in\overline{B}}f(x)\,,$$

and for arbitrary partition P of B we introduce the *upper Darboux sum*

(6b6) $$U(f,P) = \sum_{C\in P} \operatorname{vol}(C) \sup_{\overline{C}} f \,.$$

The inequality

(6b7) $$U(f,P) \le \operatorname{vol}(B) \sup_{\overline{B}} f$$

follows immediately from additivity of volume (6b2) and the trivial relation

(6b8) $$\sup_{\overline{C}} f \le \sup_{\overline{B}} f \quad \text{whenever } \overline{C} \subset \overline{B} \,.$$

In other words, we have a *box function*

(6b9) $$B \mapsto \operatorname{vol}(B) \sup_{\overline{B}} f \quad \text{for all boxes } B \subset \mathbb{R} \,,$$

and it is *superadditive:*

(6b10) $$\operatorname{vol}(B) \sup_{\overline{B}} f \ge \sum_{C\in P} \operatorname{vol}(C) \sup_{\overline{C}} f \,.$$

It follows that

(6b11) $$U(f,P) \ge U(f,P') \quad \text{whenever } P' \text{ is finer than } P \,;$$

proof: by (6b4),

(6b12) $$U(f,P') = \sum_{C\in P} U(f,P'|C) \,;$$

and

(6b13) $$U(f,P'|_C) \le \operatorname{vol}(C) \sup_{\overline{C}} f$$

by (6b7) and (6b3).

We define the *upper integral*

(6b14) $$\int_B^* f = \inf_P U(f,P) \,,$$

the infimum being taken over all partitions P of the box B. However, it may be taken only over partitions P' finer than a given partition P:

(6b15) $$\inf_{P'} U(f,P') = \inf_{P'=PP'} U(f,P') \,.$$

Proof: "$\le$" is trivial; "$\ge$": $U(f, P') \ge U(f, PP')$.

Now, a partition P' finer than P consists of its "fragments", the partitions $P'|_C$ for $C \in P$; the infimum over all such P' is the infimum over all fragments. By (6b12), $U(f, P')$ is the sum of contributions of these fragments, and we may take each infimum separately:[1]

(6b16) $$\sideset{^*}{}\int_B f = \sum_{C \in P} \sideset{^*}{}\int_C f \,,$$

which means that the upper integral is an *additive box function.*[2]

Clearly, $\sideset{^*}{}\int_B f = \sideset{^*}{}\int_B g$ whenever $f|_{\overline{B}} = g|_{\overline{B}}$. Thus, $\sideset{^*}{}\int_B f$ is well-defined for bounded $f : \overline{B} \to \mathbb{R}$, and moreover, due to 6b18 below we have

(6b17) $$\sideset{^*}{}\int_B f \text{ is well-defined for bounded } f : B^\circ \to \mathbb{R} \,.$$

6b18 Lemma. $\sideset{^*}{}\int_B f = \sideset{^*}{}\int_B g$ whenever $f|_{B^\circ} = g|_{B^\circ}$.

Proof. Given $\varepsilon > 0$, we take a partition $P = \{C_1, C_2, C_3\}$ of B in three parts such that $\overline{C}_2 \subset B^\circ$ and $\operatorname{vol}(C_2) \ge \operatorname{vol}(B) - \varepsilon$. Then

$$\begin{aligned}
\Big| \sideset{^*}{}\int_B f - \sideset{^*}{}\int_B g \Big| &= \Big| \Big(\sideset{^*}{}\int_{C_1} f + \sideset{^*}{}\int_{C_2} f + \sideset{^*}{}\int_{C_3} f \Big) - \Big(\sideset{^*}{}\int_{C_1} g + \sideset{^*}{}\int_{C_2} g + \sideset{^*}{}\int_{C_3} g \Big) \Big| \le \\
&\le \Big| \sideset{^*}{}\int_{C_1} f - \sideset{^*}{}\int_{C_1} g \Big| + \underbrace{\Big| \sideset{^*}{}\int_{C_2} f - \sideset{^*}{}\int_{C_2} g \Big|}_{=0} + \Big| \sideset{^*}{}\int_{C_3} f - \sideset{^*}{}\int_{C_3} g \Big| \le \\
&\le \Big| \sideset{^*}{}\int_{C_1} f \Big| + \Big| \sideset{^*}{}\int_{C_1} g \Big| + \Big| \sideset{^*}{}\int_{C_3} f \Big| + \Big| \sideset{^*}{}\int_{C_3} g \Big| \le \\
&\le \underbrace{\big(\operatorname{vol}(C_1) + \operatorname{vol}(C_3) \big)}_{\le \varepsilon} \Big(\sup_{\overline{B}} |f| + \sup_{\overline{B}} |g| \Big)
\end{aligned}$$

for all $\varepsilon > 0$. □

6b19 Lemma. If boxes B, C satisfy $\overline{C} \subset \overline{B}$, and $f = 0$ on $B^\circ \setminus \overline{C}$, then $\sideset{^*}{}\int_B f = \sideset{^*}{}\int_C f$.

Proof. We take a partition P of B in at most 3 parts such that $C \in P$. By 6b18, $\sideset{^*}{}\int_D f = \sideset{^*}{}\int_D 0 = 0$ for all $D \in P$, $D \ne C$. Thus, $\sideset{^*}{}\int_B f = \sum_{D \in P} \sideset{^*}{}\int_D f = \sideset{^*}{}\int_C f$. □

[1] Generally, $\inf_{x,y}(f(x) + g(y)) = \inf_x f(x) + \inf_y g(y)$ (and the same holds for more than two summands). But do not think that $\inf_x(f(x) + g(x)) = \inf_x f(x) + \inf_x g(x)$!

[2] In fact, every superadditive box function F leads to an additive box function G; $G(B) = \inf_P \sum_{C \in P} F(C)$ is the greatest additive box function satisfying $G \le F$.

It means that ${}^*\!\!\int_{\mathbb{R}} f$ is well-defined for bounded $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) \ne 0\}$ is bounded; these will be called functions with *bounded support.* Namely,[1]

(6b20) $$ {}^*\!\!\int_{\mathbb{R}} f = {}^*\!\!\int_B f \quad \text{whenever } f = 0 \text{ on } \mathbb{R} \setminus \overline{B} \,. $$

Lower Darboux sums

(6b21) $$ L(f, P) = \sum_{C \in P} \operatorname{vol}(C) \inf_{\overline{C}} f $$

may be treated similarly, with all inequalities reversed: $L(f, P) \le L(f, P')$ whenever P' is finer than P; etc. Alternatively, one may use the equality $L(f, P) = -U(-f, P)$. The *lower integral*

(6b22) $$ {}_*\!\!\int_B f = \sup_P L(f, P) $$

is another additive box function. Clearly, $L(f, P) \le U(f, P)$; it follows that $L(f, P_1) \le U(f, P_2)$, since $L(f, P_1) \le L(f, P_1P_2) \le U(f, P_1P_2) \le U(f, P_2)$. Therefore[2]

(6b23) $$ {}_*\!\!\int_B f \le {}^*\!\!\int_B f \,. $$

Also,

(6b24) $$ {}_*\!\!\int_B f = - {}^*\!\!\int_B (-f) \,. $$

If ${}_*\!\!\int_B f = {}^*\!\!\int_B f$, one says that f is *integrable* (on B), and then

(6b25) $$ {}_*\!\!\int_B f = \int_B f = {}^*\!\!\int_B f \,. $$

The same holds in a one-dimensional Euclidean affine space instead of $\mathbb{R}$. Accordingly, the integral (as well as the lower and upper integral) is invariant under translation: for every $r \in \mathbb{R}$,

(6b26) $$ \int_{[s,t]} f = \int_{[s+r,t+r]} g \quad \text{where } g(u) = f(u - r) \,, $$

and reflection:

(6b27) $$ \int_{[s,t]} f = \int_{[-t,-s]} g \quad \text{where } g(u) = f(-u) \,. $$

[1]Indeed, any two such boxes are both contained in some third such box.

[2]Generally, if $f(x) \le g(y)$ for all x, y, then $\sup_x f(x) \le \inf_y g(y)$.

6b28 Exercise. (a) If f and F satisfy (6a1) and (6a2) then ${}_*\int_B f \le F(B) \le {}^*\int_B f$, and therefore $F(B) = \int_B f$ if f is integrable. Thus, F is uniquely determined by f if f is integrable.

(b) Both $F : B \mapsto {}_*\int_B f$ and $F : B \mapsto {}^*\int_B f$ satisfy (together with f) (6a1) and (6a2). Thus, F fails to be uniquely determined by f if f is not integrable.

Formulate it accurately, and prove.

6b29 Exercise. Let

$$\begin{aligned} f(x) = 1\,, \quad g(x) = 0 \quad &\text{for all rational } x\,, \\ f(x) = 0\,, \quad g(x) = 1 \quad &\text{for all irrational } x\,. \end{aligned}$$

Prove that

$$\begin{aligned} {}_*\!\int_B (af + bg) &= \min(a,b)\,\mathrm{vol}(B)\,, \\ {}^*\!\!\int_B (af + bg) &= \max(a,b)\,\mathrm{vol}(B) \end{aligned}$$

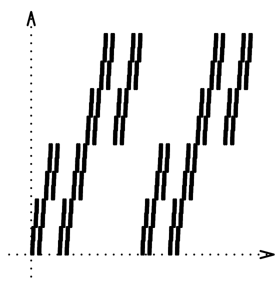
for all $a, b \in \mathbb{R}$ and all boxes B.

6b30 Exercise. Find $\int_0^1 x\,\mathrm{d}x$ using only the theory of Sect. 6b. (That is, $\int_{[0,1]} f$ where $f(t) = t$.)[1]

6b31 Exercise. Let $f : [0,1) \to [0,1)$ be defined via binary digits, by

$$f(x) = \sum_{k=1}^\infty \frac{\beta_{2k}(x)}{2^k} \quad \text{for } x = \sum_{k=1}^\infty \frac{\beta_k(x)}{2^k}\,, \; \beta_k(x) \in \{0,1\}\,, \; \liminf_k \beta_k(x) = 0\,.$$

Prove that f is integrable on $[0,1]$ and find $\int_{[0,1]} f$.[2]



[1]Hint: split $[0,1]$ into 2^k equal intervals and calculate lower and upper Darboux sums.

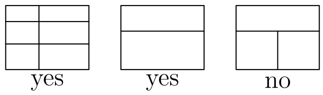
[2]Hint: split $[0,1]$ into 2^{2k} equal intervals and calculate lower and upper Darboux sums.

6c Higher dimensions

An n-dimensional *box* $B \subset \mathbb{R}^n$ is, by definition, the (Cartesian) product $B = b_1 \times \dots \times b_n$ of n one-dimensional boxes $b_1, \dots, b_n \subset \mathbb{R}$. The ($n$-dimensional) *volume* of B is, by definition, the product of lengths,

(6c1)
$$\mathrm{vol}_n(B) = \mathrm{vol}_1(b_1) \dots \mathrm{vol}_1(b_n) \, .$$

A *partition* P of B is, by definition, the product of one-dimensional partitions p_k of b_k ($k = 1, \dots, n$) in the sense that



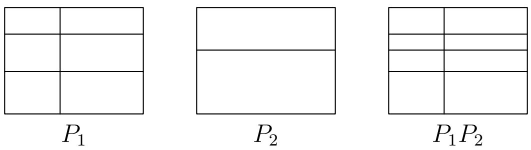
(6c2)
$$P = \{ c_1 \times \dots \times c_n : c_1 \in p_1, \dots, c_n \in p_n \} \, .$$

Additivity of volume (6b2) holds as before: $\mathrm{vol}(B) = \sum_{C \in P} \mathrm{vol}(C)$. Proof:

$$\sum_{C \in P} \mathrm{vol}(C) = \sum_{c_1 \in p_1, \dots, c_n \in p_n} \mathrm{vol}_n(c_1 \times \dots \times c_n) = \sum_{c_1 \in p_1, \dots, c_n \in p_n} \mathrm{vol}_1(c_1) \dots \mathrm{vol}_1(c_n) =$$
$$= \Big(\sum_{c_1 \in p_1} \mathrm{vol}_1(c_1) \Big) \dots \Big(\sum_{c_n \in p_n} \mathrm{vol}_1(c_n) \Big) = \mathrm{vol}_1(b_1) \dots \mathrm{vol}_1(b_n) = \mathrm{vol}(B) \, .$$

Writing (6c2) as $P = p_1 \times \dots \times p_n$ we define a *refinement* P' of P as $P' = p'_1 \times \dots \times p'_n$ where each p'_k is a refinement of p_k.

The *common refinement* of $P_1 = p_{1,1} \times \dots \times p_{1,n}$ and $P_2 = p_{2,1} \times \dots \times p_{2,n}$ is $P_1 P_2 = p_1 \times \dots \times p_n$ where each p_k is the one-dimensional common refinement $p_{1,k} p_{2,k}$.



Now the theory of Sect. 6b holds exactly as written! Just read (6b1)–(6b16) and (6b21)–(6b25) again, interpreting all boxes as n-dimensional. Lemmas 6b18, 6b19 still hold; the only change needed in their proofs is, to replace "3 parts" with "3^n parts". Thus, (6b17) and (6b20) hold.

The same applies to the product $S_1 \times \dots \times S_n$ of n one-dimensional Euclidean affine spaces instead of $\mathbb{R}^n$. Accordingly, the integral (as well as the lower and upper integrals) is invariant under translations: for every $r \in \mathbb{R}^n$,

(6c3)
$$\int_B f = \int_{B+r} g \quad \text{where } g(u) = f(u - r) \, ,$$

and reflections (of some or all the coordinates). Permutations of coordinates are also unproblematic. However, for now we cannot integrate over an arbitrary n-dimensional Euclidean space, since rotation invariance of the integral is not proved yet.

6c4 Exercise. Consider modified upper Darboux sums

$$U^\circ(f,P) = \sum_{C\in P} \operatorname{vol}(C) \sup_{C^\circ} f$$

and prove that they lead to the same upper integral:[1]

$$\inf_P U^\circ(f,P) = {}^*\!\!\int_B f\,.$$

6c5 Exercise. Let $f : [0,1)\times[0,1) \to [0,1)$ be defined by

$$f(x,y) = \sum_{k=1}^\infty \frac{\beta_k(x)}{2^{2k-1}} + \sum_{k=1}^\infty \frac{\beta_k(y)}{2^{2k}}$$

(where $\beta_k(\cdot)$ are as in 6b31). Prove that f is integrable on $[0,1]\times[0,1]$ and find $\int_{[0,1]\times[0,1]} f$.

6d Basic properties of integrals

[Sh:6.2]

The constant function $a1\!\!1 : x \mapsto a$ is integrable, and

(6d1) $$\int_B a1\!\!1 = a \operatorname{vol}(B) \quad \text{for all } a\in\mathbb{R}\,.$$

(Do not bother to use additivity of volume; just take the trivial partition P and observe that $L(f,P) = U(f,P) = a\operatorname{vol}(B)$.) Using (6b20),

(6d2) $$\int_{\mathbb{R}^n} a1\!\!1_B = a\operatorname{vol}(B) \quad \text{for every box } B\subset\mathbb{R}^n, \text{ and } a\in\mathbb{R}\,.$$

A number of properties of integrals are proved according to the pattern

(6d3) $$\begin{array}{ccccccc} \sup_C f & \longrightarrow & U(f,P) & \longrightarrow & {}^*\!\!\int_B f & \searrow & \\ & & & & & & \int_B f\,. \\ \inf_C f & \longrightarrow & L(f,P) & \longrightarrow & {}_*\!\!\int_B f & \nearrow & \end{array}$$

[1] Hint: $\operatorname{vol}(C)\sup_{C^\circ} f \ge {}^*\!\!\int_C f$.

It means: an evident property of $\sup_C f$ implies the corresponding property of $U(f,P)$ and then of ${}^*\!\int_B f$ (assuming only boundedness); similarly, from $\inf_C f$ to ${}_*\!\int_B f$; and finally, *assuming integrability,* the properties of ${}^*\!\int_B f$ and ${}_*\!\int_B f$ are combined into a property of $\int_B f$.

Properties of integrals over $\mathbb{R}^n$ follow easily from properties of integrals over boxes via (6b20). Each property below holds in two versions:

∗ all integrals are taken over a box B, and all functions are defined on B° and bounded;

∗ all integrals are taken over $\mathbb{R}^n$, and all functions are defined on $\mathbb{R}^n$, bounded, with bounded support.

(Note that bounded functions with bounded support are a vector space.[1])

Monotonicity:

(6d4) $$\text{if } f(\cdot) \le g(\cdot) \quad \text{then} \quad {}_*\!\int f \le {}_*\!\int g\,, \quad {}^*\!\int f \le {}^*\!\int g\,,$$

(6d5) $$\text{and for integrable } f,g, \quad \int f \le \int g\,.$$

(It can happen that ${}^*\!\int f > {}_*\!\int g$; find an example.)

Homogeneity:

(6d6) $${}_*\!\int cf = c\,{}_*\!\int f\,, \quad {}^*\!\int cf = c\,{}^*\!\int f \quad \text{for } c \ge 0\,;$$

(6d7) $${}_*\!\int cf = c\,{}^*\!\int f\,, \quad {}^*\!\int cf = c\,{}_*\!\int f \quad \text{for } c \le 0\,;$$

(6d8) $$\text{if } f \text{ is integrable then } cf \text{ is, and} \quad \int cf = c\int f \quad \text{for all } c \in \mathbb{R}\,.$$

(Sub-, super-) additivity:

(6d9) $${}^*\!\int (f+g) \le {}^*\!\int f + {}^*\!\int g\,;$$

(6d10) $${}_*\!\int (f+g) \ge {}_*\!\int f + {}_*\!\int g\,;$$

(6d11) $$\text{if } f,g \text{ are integrable then } f+g \text{ is, and} \quad \int (f+g) = \int f + \int g\,.$$

(It can happen that ${}^*\!\int (f+g) < {}^*\!\int f + {}^*\!\int g$; find an example.)

[1] Infinite-dimensional, of course.

Combining properties (6d8) and (6d11) we get linearity (for integrable functions only):

$$\int (c_1 f_1 + \dots + c_k f_k) = c_1 \int f_1 + \dots + c_k \int f_k \tag{6d12}$$

for $c_1, \dots, c_k \in \mathbb{R}$ and integrable $f_1, \dots, f_k$.

6d13 Exercise. Prove (6d4)–(6d12).

Translation invariance (see (6c3)): if $g(\cdot) = f(\cdot - r)$ then

$$\int_B f = \int_{B+r} g\,; \quad \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g\,; \tag{6d14}$$

and the same holds for upper and lower integrals.

6d15 Exercise. For bounded $f, g : B \to \mathbb{R}$ prove that
(a) ${}^*\!\!\int_B |fg| \le \frac12 \big({}^*\!\!\int_B f^2 + {}^*\!\!\int_B g^2 \big)$;
(b) ${}^*\!\!\int_B |fg| \le \min_{c>0} \frac12 \big(c\,{}^*\!\!\int_B f^2 + \frac1c\,{}^*\!\!\int_B g^2 \big) = \sqrt{{}^*\!\!\int_B f^2}\sqrt{{}^*\!\!\int_B g^2}$.

6d16 Exercise. (a) For f, g as in 6b29 prove that

$$\begin{gathered}
{}_*\!\!\int_B (af+b)(cg+d) = \big(\min(ad, bc) + bd \big) \operatorname{vol}(B)\,, \\
{}_*\!\!\int_B (af+b)^2 = \min\big((a+b)^2, b^2 \big) \operatorname{vol}(B)\,, \\
{}_*\!\!\int_B (cg+d)^2 = \min\big((c+d)^2, d^2 \big) \operatorname{vol}(B)
\end{gathered}$$

for all $a, b, c, d \in \mathbb{R}$ and all intervals B.
(b) Prove existence of bounded $f, g : B \to \mathbb{R}$ such that ${}_*\!\!\int_B |fg| > \sqrt{{}_*\!\!\int_B f^2}\sqrt{{}_*\!\!\int_B g^2}$.

6d17 Exercise. For given $s_1, \dots, s_n > 0$ define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(t_1, \dots, t_n) = (s_1 t_1, \dots, s_n t_n)$. Prove that

$$s_1 \dots s_n \,{}_*\!\!\int f \circ T = {}_*\!\!\int f\,, \quad s_1 \dots s_n \,{}^*\!\!\int f \circ T = {}^*\!\!\int f$$

(for bounded f with bounded support); if f is integrable, then $f \circ T$ is integrable and

$$s_1 \dots s_n \int_{\mathbb{R}^n} f \circ T = \int_{\mathbb{R}^n} f\,.$$

6e Normed space of equivalence classes

Let $B \subset \mathbb{R}^n$ be a box. All bounded functions $B^\circ \to \mathbb{R}$ are a vector space.[1] On this space, the functional[2]

$$f \mapsto {}^*\!\!\int_B |f|$$

is a *seminorm;* that is, satisfies the first two conditions of 1e3(a),

$$\begin{gathered} {}^*\!\!\int_B |cf| = |c| \, {}^*\!\!\int_B |f| \, , \\ {}^*\!\!\int_B |f+g| \le {}^*\!\!\int_B |f| + {}^*\!\!\int_B |g| \end{gathered}$$

(think, why), but violates the third condition,

$$ {}^*\!\!\int_B |f| > 0 \quad \text{whenever } f \ne 0 \, . \qquad \text{(Wrong!)}$$

Functions f such that ${}^*\!\!\int_B |f| = 0$ will be called *negligible.* Functions f, g such that $f - g$ is negligible will be called *equivalent.* For example, for each box B functions $1\!\!1_{B^\circ}$, $1\!\!1_B$ and $1\!\!1_{\overline{B}}$ are equivalent, see (6d2). The equivalence class of f will be denoted $[f]$.

6e1 Exercise. (a) Negligible functions are an infinite-dimensional vector space.

(b) Equivalence classes are an infinite-dimensional vector space; the functional

$$[f] \mapsto {}^*\!\!\int_B |f|$$

is well-defined on this vector space, and is a norm.[3]

Prove it.

Thus, equivalence classes are a normed space, therefore also a metric space:

$$\rho\big([f],[g]\big) = \| \, [f] - [g] \, \| = {}^*\!\!\int_B |f-g| \, ;$$

this metric will be called the *integral metric,* and the corresponding convergence the *integral convergence.*

---

[1] Infinite-dimensional, of course.

[2] Functions on infinite-dimensional spaces are often called functionals.

[3] In fact, every seminorm on a vector space leads to a normed space of equivalence classes.

6e2 Exercise. Functionals

$$[f] \mapsto {}_*\!\!\int_B f \,, \quad [f] \mapsto {}^*\!\!\int_B f$$

on the normed space of equivalence classes are well-defined and continuous; moreover,

$$\Big| {}_*\!\!\int_B f - {}_*\!\!\int_B g \Big| \le \| f - g \| \,, \quad \Big| {}^*\!\!\int_B f - {}^*\!\!\int_B g \Big| \le \| f - g \| \,.$$

Prove it.

Here and henceforth we often write $\|f\|$ instead of $\| \, [f] \, \|$.

6e3 Exercise. (a) A function equivalent to an integrable function is integrable;

(b) equivalence classes of integrable functions are a closed set in the normed space of equivalence classes,[1] and the functional $[f] \mapsto \int_B f$ on this set is continuous.

Prove it.[2]

6e4 Exercise. (a) Uniform convergence of functions implies integral convergence; prove it;

(b) the converse is wrong; find a counterexample.

6e5 Remark. Pointwise convergence does not imply integral convergence, even if the functions are uniformly bounded.[3] Here is a counterexample. We take a sequence $(x_k)_k$ of pairwise different points $x_k \in (0,1)$ that is dense in $(0,1)$ and consider dense countable sets $A_k = \{x_{k+1}, x_{k+2}, \dots\}$. Clearly, $A_1 \supset A_2 \supset \dots$ and $\bigcap_k A_k = \emptyset$. Indicator functions $f_k = 1\!\!1_{A_k}$ converge to 0 pointwise (and monotonically). Nevertheless, ${}^*\!\!\int_{(0,1)} f_k = 1$ for all k.

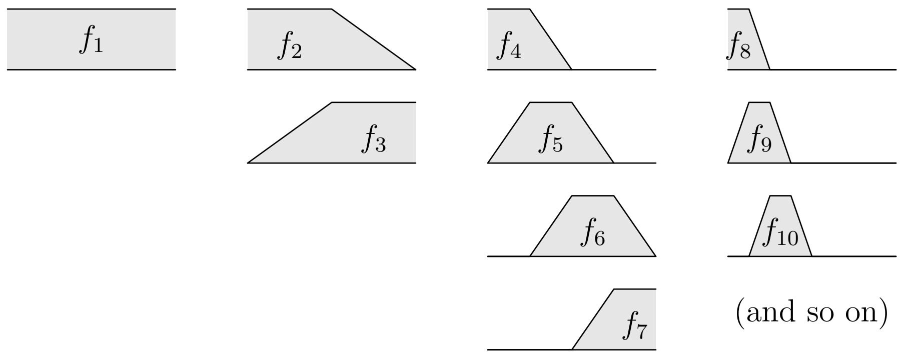
6e6 Remark. Integral convergence does not imply pointwise convergence, even if the functions are continuous. Not even in "most" of the points. Here

[1] Two dense subsets of this closed set are treated in Sect. 6f: continuous functions, and step functions. (Or rather, their equivalence classes.)

[2] Hint: use 6e2.

[3] It does, if the functions are integrable! But this fact is far beyond basis of integration. We'll return to it later.

is a counterexample on $B = (0,1) \subset \mathbb{R}$:



6f Sandwiching a function

Given a partition P of a box B, and a bounded function $f : B \to \mathbb{R}$, we introduce "step functions"

$$f_P^-(x) = \min_{C \in P: x \in \overline{C}} \inf_{\overline{C}} f \,, \quad f_P^+(x) = \max_{C \in P: x \in \overline{C}} \sup_{\overline{C}} f$$

and note that

$$f_P^-(\cdot) \le f(\cdot) \le f_P^+(\cdot)$$

and

$$\int_B f_P^- = L(f,P) \,, \quad \int_B f_P^+ = U(f,P) \,;$$

the latter follows from 6c4, since $U^\circ(f_P^+, P) = U(f,P)$ (and the same holds for the modified lower Darboux sum $L^\circ(f_P^+, P)$). By (6b14) and (6b22),

$$\sup_P \int_B f_P^- = {}_*\!\int_B f \,, \quad \inf_P \int_B f_P^+ = {}^*\!\!\int_B f \,.$$

If f is integrable then for every $\varepsilon > 0$ we can sandwich f between ε-close step functions (for any given $\varepsilon > 0$):

$$f_P^-(\cdot) \le f(\cdot) \le f_P^+(\cdot) \,, \quad \int_B (f_P^+ - f_P^-) \le \varepsilon \,.$$

If f is uniformly continuous on B then for every $\varepsilon > 0$ there exists P such that $f_P^+(\cdot) - f_P^-(\cdot) \le \varepsilon$ on B (think, why), therefore ${}^*\!\int_B f - {}_*\!\int_B f \le \varepsilon$, which shows that

(6f1) every uniformly continuous function on a box is integrable.

6f2 Exercise. Every continuous function on a box is integrable.
Prove it.[1]

What about sandwiching an integrable function between continuous functions?

6f3 Lemma. For every box $B \subset \mathbb{R}^n$ and every $\varepsilon > 0$ there exist continuous functions $g, h : \mathbb{R}^n \to [0,1]$ with bounded support such that $g(\cdot) \le 1\!\!1_B(\cdot) \le h(\cdot)$ and ${}^*\!\!\int_{\mathbb{R}^n} (h - g) \le \varepsilon$.

Proof. We take a continuous $h : \mathbb{R}^n \to [0,1]$ such that $h(\cdot) = 1$ on B and $h(\cdot) = 0$ outside the δ-neighborhood $B_{+\delta} = \{x : \operatorname{dist}(x, B) \le \delta\}$.[2] Similarly we take a continuous $g : \mathbb{R}^n \to [0,1]$ such that $g(\cdot) = 0$ outside B, and $g(\cdot) = 1$ on $B_{-\delta} = \{x : \operatorname{dist}(x, \mathbb{R}^n \setminus B) \ge \delta\}$. We have ${}^*\!\!\int_{\mathbb{R}^n} (h - g) \le \int_{\mathbb{R}^n} (1\!\!1_{B_{+\delta}} - 1\!\!1_{B_{-\delta}}) = \operatorname{vol}(B_{+\delta}) - \operatorname{vol}(B_{-\delta}) \le \varepsilon$ if δ is small enough.[3] $\square$

We see that $1\!\!1_B$ can be sandwiched between continuous functions (and by continuity, the inequality $g(\cdot) \le 1\!\!1_B(\cdot) \le h(\cdot)$ implies $g(\cdot) \le 1\!\!1_{B^\circ}(\cdot) \le 1\!\!1_{\overline{B}}(\cdot) \le h(\cdot)$). The same holds for $a1\!\!1_B$ for arbitrary $a \in \mathbb{R}$ (think, what happens for $a < 0$).

6f4 Exercise. Prove that $\| \max(f_1, f_2) - \max(g_1, g_2) \| \le \|f_1 - g_1\| + \|f_2 - g_2\|$ for all bounded $f_1, f_2, g_1, g_2 : B \to \mathbb{R}$.[4] (Pointwise maxima are meant.)

If f_1 and f_2 can be sandwiched between continuous functions, then, by 6f4, also $\max(f_1, f_2)$ can be so sandwiched. The same holds for $f_1, \dots, f_k$. Therefore, step functions f_P^-, f_P^+ can be so sandwiched. Using h for f_P^+, g for f_P^-, and taking 6e2 into account, we get the following result.

6f5 Proposition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support, and $\varepsilon > 0$. Then there exist continuous $g, h : \mathbb{R}^n \to \mathbb{R}$ with bounded support such that

$$g(\cdot) \le f(\cdot) \le h(\cdot) \, , \quad \int_{\mathbb{R}^n} (h - g) \le \varepsilon + {}^*\!\!\int_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} f \, .$$

And, of course,

$$\int_{\mathbb{R}^n} g \ge -\varepsilon + {}_*\!\!\int_{\mathbb{R}^n} f \, , \quad \int_{\mathbb{R}^n} h \le \varepsilon + {}^*\!\!\int_{\mathbb{R}^n} f \, . \tag{6f6}$$

[1]Hint: similarly to the proof of 6b18, take a smaller box $\overline{C} \subset B^\circ$, apply (6f1) to $f|_{\overline{C}}$, and use 6e3(b).

[2]There are many ways to do so. One way: do it first in one dimension, and then take the product. Moreover, $h \in C^1(\mathbb{R}^n)$ can be chosen.

[3]Indeed, $(t_1 - s_1 + 2\delta) \dots (t_n - s_n + 2\delta) - (t_1 - s_1 - 2\delta) \dots (t_n - s_n - 2\delta) \to 0$ as $\delta \to 0$.

[4]Hint: $|\max(a_1, a_2) - \max(b_1, b_2)| \le \max(|a_1 - b_1|, |a_2 - b_2|)$ for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

6f7 Corollary. Continuous functions are dense among integrable functions (in the integral metric).

6f8 Exercise. (a) Binary operations

$$[f], [g] \mapsto [\min(f, g)] , \quad [f], [g] \mapsto [\max(f, g)]$$

(pointwise minimum and maximum, denoted also $f \wedge g$ and $f \vee g$) on the normed space of equivalence classes are well-defined and continuous.

(b) If f and g are integrable then $\min(f, g)$ and $\max(f, g)$ are integrable.

Prove it.[1]

6f9 Exercise. (a) Pointwise multiplication

$$[f], [g] \mapsto [fg]$$

on the normed space of equivalence classes is well-defined; it is continuous on the subset of functions $B \to [-1, 1]$.

(b) If f and g are integrable then fg is integrable.

Prove it.[2]

6f10 Remark. The multiplication is also continuous on the subset of functions $B \to [-M, M]$ for every M, but not on the whole space.[3] A counterexample: $B = (0, 1) \subset \mathbb{R}$, $f_k = g_k = k\mathbb{1}_{(0,1/k^2)}$; then $\|f_k\| = \|g_k\| = 1/k \to 0$, but $\|f_k g_k\| = 1$ for all k.

6g Volume as Jordan measure

[Sh:6.5]

The indicator function $\mathbb{1}_E$ of a bounded set $E \subset \mathbb{R}^n$ evidently is a bounded function with bounded support.

6g1 Definition. Let $E \subset \mathbb{R}^n$ be a bounded set. Its *inner Jordan measure* $v_*(E)$ and *outer Jordan measure* $v^*(E)$ are

$$v_*(E) = {}_*\!\!\int_{\mathbb{R}^n} \mathbb{1}_E \, , \quad v^*(E) = {}^*\!\!\int_{\mathbb{R}^n} \mathbb{1}_E \, .$$

[1]Hint: (a) use 6f4; (b) approximation (or sandwich).

[2]Hint: (a) $\|f_1 g_1 - f_2 g_2\| \le \|f_1 - f_2\| + \|g_1 - g_2\|$ for functions $B \to [-1, 1]$; (b) WLOG, $f, g : B \to [-1, 1]$.

[3]Even though the whole space is the union of these subsets! You see, these subsets have no interior points.

If they are equal (that is, if $\mathbb{1}_E$ is integrable) then E is *Jordan measurable,*[1] and its Jordan measure[2] is

$$v(E) = \int_{\mathbb{R}^n} \mathbb{1}_E \, .$$

By (6d2), every box $B \subset \mathbb{R}^n$ is Jordan measurable, and

(6g2) $$v(B) = \operatorname{vol}(B) \, .$$

Therefore its boundary $\partial B = \overline{B} \setminus B^\circ$ is Jordan measurable, and

(6g3) $$v(\partial B) = 0 \, ,$$

since $\mathbb{1}_{\partial B} = \mathbb{1}_{\overline{B}} - \mathbb{1}_{B^\circ}$ and $\int \mathbb{1}_{\overline{B}} = \int \mathbb{1}_{B^\circ} = \operatorname{vol}(B)$.

Monotonicity (follows from (6d4)):

(6g4) $$E_1 \subset E_2 \quad \text{implies} \quad v_*(E_1) \le v_*(E_2) \, , \; v^*(E_1) \le v^*(E_2) \, .$$

(Sub-, super-) additivity (follows from (6d9), (6d10), (6d11) and (6d4)):

(6g5) $$v^*(E_1 \cup E_2) \le v^*(E_1) + v^*(E_2) \, ,$$

(6g6) $$v_*(E_1 \uplus E_2) \ge v_*(E_1) + v_*(E_2) \, ;$$

(6g7) if E_1, E_2 are Jordan measurable then $E_1 \uplus E_2$ is, and
$$v(E_1 \uplus E_2) = v(E_1) + v(E_2) \, .$$

Here "$\uplus$" stands for disjoint union; that is, $A \uplus B$ is just $A \cup B$ but only if $A \cap B = \emptyset$ (otherwise undefined). Thus, disjointedness is assumed in (6g6), (6g7), and implies $\mathbb{1}_{E_1 \uplus E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2}$.

Translation invariance (follows from (6d14)): for every $r \in \mathbb{R}^n$,

(6g8) $$v_*(E + r) = v_*(E) \, , \quad v^*(E + r) = v^*(E) \, .$$

We define a *set of volume zero* as a bounded set $E \subset \mathbb{R}^n$ such that $v^*(E) = 0$. Equivalently: a Jordan measurable set such that $v(E) = 0$. Due to (6g4), (6g5),

(6g9) if $E \subset F$ and F is of volume zero then E is;

(6g10) if $E_1, \dots, E_k$ are of volume zero then $E_1 \cup \dots \cup E_k$ is.

[1] Or just a Jordan set.

[2] Or the n-dimensional volume, or Jordan content, or Peano-Jordan measure, etc.

6g11 Exercise. Prove that

(a) the inner Jordan measure of an open ball of radius r in $\mathbb{R}^n$ is $\ge \big(\frac{2r}{\sqrt{n}}\big)^n$;

(b) every set of volume zero has empty interior.

6g12 Exercise. For $s_1, \dots, s_n$ and T as in 6d17 prove that

$$v_*(T(E)) = s_1 \dots s_n v_*(E) \, , \quad v^*(T(E)) = s_1 \dots s_n v^*(E)$$

for every bounded E, and if E is Jordan measurable then $T(E)$ is Jordan measurable and $v(T(E)) = s_1 \dots s_n v(E)$. In particular, $v(sE) = s^n v(E)$.

6g13 Exercise. If bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support differ only on a set of volume zero then they are equivalent.

Prove it.[1] [2]

We may safely ignore values of integrands on sets of volume zero (as far as they are bounded). Likewise we may ignore sets of volume zero when dealing with Jordan measure.

We may add "outside a set of volume zero" to (6d4)–(6d12), like this:

Monotonicity: if $f(\cdot) \le g(\cdot)$ outside a set of volume zero then

$$\int_* f \le \int_* g \, , \quad \int^* f \le \int^* g \, , \tag{6g14}$$

$$\text{and for integrable } f, g, \quad \int f \le \int g \, . \tag{6g15}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, and $E \subset \mathbb{R}^n$ a Jordan set. By 6f9(b), $f \cdot 1\!\!1_E$ is integrable, and we define

$$\int_E f = \int_{\mathbb{R}^n} f \cdot 1\!\!1_E \, . \tag{6g16}$$

Similarly to (6g7),

$$\int_{E_1 \uplus E_2} f = \int_{E_1} f + \int_{E_2} f \, . \tag{6g17}$$

Thus, the additive box function $B \mapsto \int_B f$ is extended to an additive set function $E \mapsto \int_E f$. If $v(E) = 0$ then $\int_E f = 0$ by 6g13. Otherwise

$$\int_E f = \int_E a \quad \text{where} \quad a = \frac{1}{v(E)} \int_E f \, ; \tag{6g18}$$

this a is called the *mean* (value) of f on E. Note that $a \in [\inf_E f, \sup_E f]$. Values of f outside E being irrelevant, integrability on E and integral on E are well-defined for $f : E \to \mathbb{R}$.

[1] Hint: $|f - g| \le \text{const} \cdot 1\!\!1_E$.

[2] "Sets of volume zero are small enough that they don't interfere with integration" [Sh:p.272].

6h The area under a graph

6h1 Proposition. Let $f : B \to [0, \infty)$ be an integrable function on a box $B \subset \mathbb{R}^n$, and

$$E = \{(x,t) : x \in B,\, 0 \le t \le f(x)\} \subset \mathbb{R}^{n+1} .$$

Then E is Jordan measurable (in $\mathbb{R}^{n+1}$), and

$$v(E) = \int_B f .$$

Proof. Let P be a partition of B. Consider sets

$$E_- = \bigcup_{C \in P} C \times [0, \inf_C f] , \quad E_+ = \bigcup_{C \in P} C \times [0, \sup_C f] .$$

We have $E_- \subset E \subset E_+$. The set E_+ is a finite union of boxes (in $\mathbb{R}^{n+1}$), disjoint up to sets of volume zero; by (6g7), E_+ is Jordan measurable, and $v(E_+)$ is the sum of the volumes of these boxes; the same holds for E_-; namely,

$$v(E_-) = L(f, P) , \quad v(E_+) = U(f, P) .$$

The relation $E_- \subset E \subset E_+$ implies $v(E_-) \le v_*(E) \le v^*(E) \le v(E_+)$, thus $L(f,P) \le v_*(E) \le v^*(E) \le U(f,P)$, which implies ${}_*\!\int_B f \le v_*(E) \le v^*(E) \le {}^*\!\int_B f$. The rest is evident. □

6h2 Exercise. For f and B as in 6h1, the graph

$$\Gamma = \{(x, f(x)) : x \in B\} \subset \mathbb{R}^{n+1}$$

is of volume zero.

Prove it.[1]

6h3 Exercise. Prove that

(a) the disk $\{x : |x| \le 1\} \subset \mathbb{R}^2$ is Jordan measurable;

(b) the ball $\{x : |x| \le 1\} \subset \mathbb{R}^n$ is Jordan measurable;

(c) for every $p > 0$ the set $E_p = \{(x_1, \dots, x_n) : |x_1|^p + \dots + |x_n|^p \le 1\} \subset \mathbb{R}^n$ is Jordan measurable;

(d) $v(E_p)$ is a strictly increasing function of p.

6h4 Exercise. For the balls $E_r = \{x : |x| \le r\} \subset \mathbb{R}^n$ prove that

(a) $v(E_r) = r^n v(E_1)$;

(b) $v(E_r) < \mathrm{e}^{-n(1-r)} v(E_1)$ for $r < 1$.

A wonder: in high dimension the volume of a ball concentrates near the sphere!

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[1]Hint: maybe, $\Gamma \subset E_+ \setminus E_-$?

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