9 Change of variables

Change of variables is the most powerful tool for calculating multidimensional integrals. Two kinds of differentiation are instrumental: of mappings (treated in Sections 2–5) and of set functions (treated in Sect. 8c).

9a What is the problem

The area of a disk $\{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ may be calculated by iterated integral,

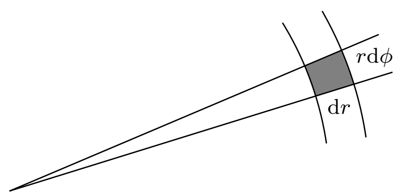
$$\int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y = \int_{-1}^{1} 2\sqrt{1-x^2}\, \mathrm{d}x = \dots$$

or alternatively, in polar coordinates,

$$\int_0^1 r\, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi = \int_0^1 2\pi r\, \mathrm{d}r = \pi\,;$$

the latter way is much easier! Note "$r\mathrm{d}r$" rather than "$\mathrm{d}r$" (otherwise we would get 2π instead of π).

Why the factor r? In analogy to the one-dimensional theory we may expect something like $\frac{\mathrm{d}x\,\mathrm{d}y}{\mathrm{d}r\,\mathrm{d}\varphi}$; is it r? Well, basically, it is r because an infinitesimal rectangle $[r, r+\mathrm{d}r] \times [\varphi, \varphi+\mathrm{d}\varphi]$ of area $\mathrm{d}r \cdot \mathrm{d}\varphi$ on the (r, φ)-plane corresponds to an infinitesimal rectangle or area $\mathrm{d}r \cdot r\mathrm{d}\varphi$ on the (x, y)-plane.



9a1 Theorem. Let $U, V \subset \mathbb{R}^n$ be Jordan measurable open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$ a bounded function such that the function $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$ is also bounded. Then

(a) f is integrable on V if and only if $(f \circ \varphi)|\det D\varphi|$ is integrable on U; and

(b) if they are integrable, then

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \,.$$

9a2 Exercise. Prove that the following is an equivalent reformulation of Theorem 9a1.[1]

Let $U, V \subset \mathbb{R}^n$ be Jordan measurable open sets, $\varphi : V \to U$ a diffeomorphism, and $f : U \to \mathbb{R}$ a bounded function such that the function $f|\det D(\varphi^{-1})| : U \to \mathbb{R}$ is also bounded. Then

(a) $f \circ \varphi$ is integrable on V if and only if $f|\det D(\varphi^{-1})|$ is integrable on U; and

(b) if they are integrable, then

$$\int_V f \circ \varphi = \int_U f|\det D(\varphi^{-1})| \,.$$

9b Examples

In this section we take for granted Theorem 9a1 (to be proved in Sect. 9h).

9b1 Exercise. Show that 7d4 and 7d5 are special cases of 9a1.

9b2 Exercise (polar coordinates in $\mathbb{R}^2$)**.** (a) Prove that

$$\int_{x^2+y^2<R^2} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{0<r<R,0<\theta<2\pi} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\mathrm{d}\theta$$

for every integrable function f on the disk $x^2 + y^2 < R^2$;[2]

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[1] Hint:

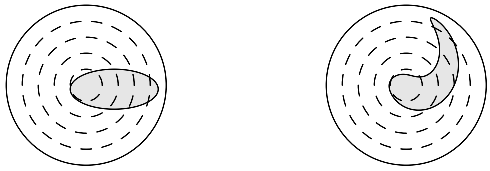
$$\begin{tikzcd} U \arrow[d, "\varphi"'] \arrow[dr, "f\circ\varphi"] & \\ V \arrow[r, "f"'] & \mathbb{R} \end{tikzcd} \quad ; \text{ redraw: } \quad \begin{tikzcd} V \arrow[d, "\varphi^{-1}"'] \arrow[dr, "f"] & \\ U \arrow[r, "f\circ\varphi"'] & \mathbb{R} \end{tikzcd} \quad ; \text{ relabel: } \quad \begin{tikzcd} V \arrow[d, "\varphi"'] \arrow[dr, "f\circ\varphi"] & \\ U \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

[2] Do you use a diffeomorphism between $(0, R) \times (0, 2\pi)$ and the disk? (Look closely!)

(b) it can happen that the function $(r, \theta) \mapsto rf(r\cos\theta, r\sin\theta)$ is integrable on $(0, R) \times (0, 2\pi)$, but f is not integrable on the disk; find a counterexample;

(c) however, (b) cannot happen if f is bounded on the disk; prove it.[1]

In particular, we have now the "curvilinear Cavalieri principle for concentric circles" promised in 7d13.



9b3 Exercise (spherical coordinates in $\mathbb{R}^3$)**.** Consider the mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$, $\Psi(r, \varphi, \theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$.

(a) Draw the images of the planes $r = \text{const}$, $\varphi = \text{const}$, $\theta = \text{const}$, and of the lines $(\varphi, \theta) = \text{const}$, $(r, \theta) = \text{const}$, $(r, \varphi) = \text{const}$.

(b) Show that Ψ is surjective but not injective.

(c) Show that $|\det D\Psi| = r^2 \sin\theta$. Find the points (r, φ, θ), where the operator $D\Psi$ is invertible.

(d) Let $V = (0, \infty) \times (-\pi, \pi) \times (0, \pi)$. Prove that $\Psi|_V$ is injective. Find $U = \Psi(V)$.

9b4 Exercise. Compute the integral $\iiint_{x^2+y^2+(z-2)^2 \le 1} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2+z^2}$.

Answer: $\pi\left(2 - \frac{3}{2}\log 3\right)$. [2]

9b5 Exercise. Compute the integral $\iint \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$. [3]

9b6 Exercise. Compute the integral over the four-dimensional unit ball: $\iiiint_{x^2+y^2+u^2+v^2 \le 1} e^{x^2+y^2-u^2-v^2}\, \mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v$. [4]

9b7 Exercise. Compute the integral $\iiint |xyz|\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the ellipsoid $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1\}$.

Answer: $\frac{a^2b^2c^2}{6}$. [5]

---

[1] Do not forget: Theorem 9a1 is taken for granted.

[2] Hint: $1 < r < 3$; $\cos\theta > \frac{r^2+3}{4r}$.

[3] Hints: use polar coordinates; $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$; $0 < r < \sqrt{\cos 2\varphi}$; $1 + \cos 2\varphi = 2\cos^2\varphi$; $\int \frac{\mathrm{d}\varphi}{\cos^2\varphi} = \tan\varphi$.

[4] Hint: The integral equals $\iint_{x^2+y^2\le 1} e^{x^2+y^2} \left(\iint_{u^2+v^2 \le 1-(x^2+y^2)} e^{-(u^2+v^2)}\, \mathrm{d}u\mathrm{d}v\right) \mathrm{d}x\mathrm{d}y$. Now use the polar coordinates.

[5] Hint: 6d17 can help.

The *centroid*[1] of a Jordan measurable set $E \subset \mathbb{R}^n$ of non-zero volume is the point $C_E \in \mathbb{R}^n$ such that for every linear (or affine) $f : \mathbb{R}^n \to \mathbb{R}$ the mean of f on E (recall (6g18)) is equal to $f(C_E)$. That is,

$$C_E = \frac{1}{v(E)} \biggl(\int_E x_1 \, \mathrm{d}x, \dots, \int_E x_n \, \mathrm{d}x \biggr) ,$$

which is often abbreviated to $C_E = \frac{1}{v(E)} \int_E x \, \mathrm{d}x$.

9b8 Exercise. Find the centroids of the following bodies in $\mathbb{R}^3$:

(a) The cone $\{(x, y, z) : h\sqrt{x^2 + y^2} < z < h\}$ for a given $h > 0$.

(b) The tetrahedron bounded by the three coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(c) The hemispherical shell $\{a^2 \le x^2 + y^2 + z^2 \le b^2, \ z \ge 0\}$.

(d) The octant of the ellipsoid $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1, \ x, y, z \ge 0\}$.

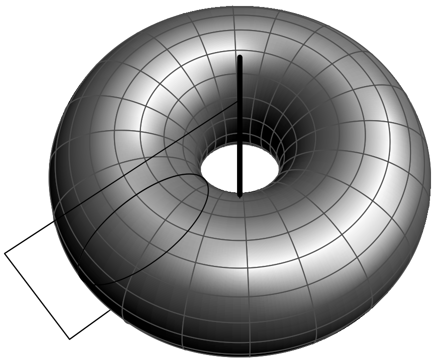
The *solid torus* in $\mathbb{R}^3$ with minor radius r and major radius R (for $0 < r < R < \infty$) is the set

$$\tilde{\Omega} = \bigl\{(x, y, z) : \bigl(\sqrt{x^2 + y^2} - R\bigr)^2 + z^2 \le r^2\bigr\} \subset \mathbb{R}^3$$

generated by rotating the disk

$$\Omega = \bigl\{(x, z) : (x - R)^2 + z^2 \le r^2\bigr\} \subset \mathbb{R}^2$$

on the (x, z) plane (with the center $(R, 0)$ and radius r) about the z axis.



Interestingly, the volume $2\pi^2 R r^2$ of $\tilde{\Omega}$ is equal to the area πr^2 of Ω multiplied by the distance $2\pi R$ traveled by the center of Ω. (Thus, it is also equal to the volume of the cylinder $\{(x, y, z) : (x, z) \in \Omega, \ y \in [0, 2\pi R].\}$) Moreover, this is a special case of a general property of all solids of revolution.

[1]In other words, the barycenter of (the uniform distribution on) E.

9b9 Proposition (the second Pappus's centroid theorem)**.** [1] [2] Let $\Omega \subset (0,\infty)\times\mathbb{R} \subset \mathbb{R}^2$ be a Jordan measurable set and $\tilde\Omega = \{(x,y,z) : \big(\sqrt{x^2+y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde\Omega$ is Jordan measurable, and

$$v_3(\tilde\Omega) = v_2(\Omega)\cdot 2\pi x_{C_\Omega}\,;$$

here $C_\Omega = (x_{C_\Omega}, z_{C_\Omega})$ is the centroid of Ω.

9b10 Exercise. Prove Prop. 9b9.[3]

9c Rotation invariance

9c1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry,[4] and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded function with bounded support. Then

(a) $$\,_*\!\!\int_{\mathbb{R}^n} f\circ T = \,_*\!\!\int_{\mathbb{R}^n} f\,, \qquad \int^*_{\mathbb{R}^n} f\circ T = \int^*_{\mathbb{R}^n} f\,;$$

(b) $f\circ T$ is integrable if and only if f is integrable, and in this case

$$\int_{\mathbb{R}^n} f\circ T = \int_{\mathbb{R}^n} f\,.$$

9c2 Corollary. (a) $v_*\big(T(E)\big) = v_*(E)$ and $v^*\big(T(E)\big) = v^*(E)$ for all bounded $E \subset \mathbb{R}^n$;

(b) $T(E)$ is Jordan measurable if and only if E is, and then $v\big(T(E)\big) = v(E)$.

In order to prove Th. 9c1 we need $v\big(T(B)\big) = v(B)$ for every box $B \subset \mathbb{R}^n$. But first we need Jordan measurability of $T(B)$. There are several ways to prove it. Here is one of them (not the most elementary).[5]

9c3 Lemma. For every norm $\|\cdot\|$ on $\mathbb{R}^n$, the set $\{x : \|x\| = 1\}$ is of volume zero, and the sets $\{x : \|x\| < 1\}$, $\{x : \|x\| \le 1\}$ are Jordan measurable.

[1]Pappus of Alexandria ($\approx$ 0290–0350) was one of the last great Greek mathematicians of Antiquity.

[2]The first Pappus's centroid theorem, about the surface area, has to wait for Analysis 4.

[3]Hint: use cylindrical coordinates: $\Psi(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z)$.

[4]That is, a linear bijection satisfying $\forall x \;\; |Tx| = |x|$. Note that T^{-1} is also a linear isometry.

[5]Do you see a more elementary way?

Proof. We denote $E = \{x : \|x\| \le 1\}$. It follows easily from 1e7 that $E^\circ = \{x : \|x\| < 1\}$, $\overline{E} = E$, and $\partial E = \{x : \|x\| = 1\} = \partial(E^\circ)$. By (8e3), $v^*\big((1-\varepsilon)E\big) \le v_*(E^\circ)$ for all $\varepsilon \in (0,1)$. By 6g12, $v^*\big((1-\varepsilon)E\big) = (1-\varepsilon)^n v^*(E)$. By 8b7(b), $v_*(E^\circ) = v_*(E)$. We get $(1-\varepsilon)^n v^*(E) \le v_*(E)$ for all $\varepsilon \in (0,1)$, therefore $v^*(E) \le v_*(E)$, that is, E is Jordan measurable. By 8d4(b), $\partial B = \partial(B^\circ)$ is of volume zero, and E° is Jordan measurable. $\square$

9c4 Exercise. (a) For every box $B \subset \mathbb{R}^n$ there exist $x_0 \in \mathbb{R}^n$ and a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $B^\circ = \{x : \|x - x_0\| < 1\}$ and $\overline{B} = \{x : \|x - x_0\| \le 1\}$.

(b) For every box $B \subset \mathbb{R}^n$ and every invertible linear $T : \mathbb{R}^n \to \mathbb{R}^n$, the set $T(B)$ is Jordan measurable.

Prove it.

We return to a linear isometry T.[1]

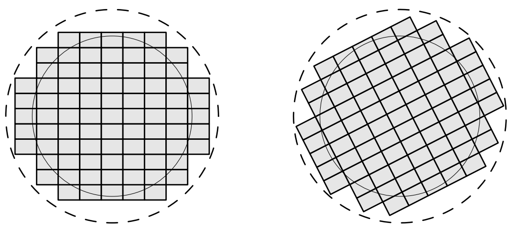
9c5 Lemma. $v\big(T(B)\big) = v(B)$ for every box $B \subset \mathbb{R}^n$.

Proof. WLOG, the box is centered: $\overline{B} = \{x : \|x\| \le 1\}$ where $\|x\| = 2\max\big(\frac{|x_1|}{c_1}, \dots, \frac{|x_n|}{c_n}\big)$; $v(B) = c_1 \dots c_n$. For arbitrary $R \in (0,\infty)$ we introduce the set $J_R \subset \mathbb{Z}^n$ of all $(i_1, \dots, i_n)$ such that the shifted box $\overline{B} + ic$ intersects the ball $RD = \{x : |x| \le R\}$; here ic stands for $(i_1 c_1, \dots, i_n c_n)$. We have $RD \subset \cup_{i \in J_R} (\overline{B} + ic)$, therefore

$$v(RD) \le \#(J_R) v(B) \,.$$

Also, $T(RD) \subset T\big(\cup_{i \in J_R}(\overline{B} + ic)\big)$; taking into account that T is isometric we have $RD \subset \cup_{i \in J_R}(T(\overline{B}) + T(ic))$, therefore

$$v(RD) \le \#(J_R) v\big(T(B)\big) \,.$$



On the other hand, taking $r \in (0,\infty)$ such that $B \subset rD$, we have $\cup_{i \in J_R}(\overline{B} + ic) \subset (R+2r)D$ (think, why); and the interiors $B^\circ + ic$ are pairwise disjoint; therefore

$$\#(J_R) v(B) \le v\big((R+2r)D\big) \,.$$

Applying T as before we get

$$\#(J_R) v\big(T(B)\big) \le v\big((R+2r)D\big) \,.$$

[1] See also 7d14.

It follows that
$$\frac{R^n}{(R+2r)^n} \le \frac{v\big(T(B)\big)}{v(B)} \le \frac{(R+2r)^n}{R^n}$$
for all R (while r is constant); it remains to take $R \to \infty$. $\square$

***Proof** of Th. 9c1.* Item (b) for the indicator $f = 1\!\!1_B$ of a box B follows from 9c5, since $f \circ T = 1\!\!1_{T^{-1}(B)}$ (and T^{-1} also is a linear isometry). By linearity, Item (b) holds for all step functions f.

In general, given f and $\varepsilon > 0$, we take a step function h such that $h \ge f$ and $\int h \le {}^*\!\!\int f + \varepsilon$, note that $h \circ T \ge f \circ T$ and get
$$\overset{*}{\int} f \circ T \le \overset{*}{\int} h \circ T = \int h \circ T = \int h \le \overset{*}{\int} f + \varepsilon$$
for all $\varepsilon > 0$; that is, ${}^*\!\!\int f \circ T \le {}^*\!\!\int f$.

Also, ${}^*\!\!\int f = {}^*\!\!\int (f \circ T) \circ T^{-1} \le {}^*\!\!\int f \circ T$; thus, ${}^*\!\!\int f \circ T = {}^*\!\!\int f$. Similarly (or taking $(-f)$), ${}_*\!\!\int f \circ T = {}_*\!\!\int f$, which completes the proof of Item (a). Item (b) follows immediately. $\square$

Given an n-dimensional Euclidean vector space E, we choose a linear isometry $E \to \mathbb{R}^n$ and transfer the Riemann integral (and the Jordan measure) from $\mathbb{R}^n$ to E. By Theorem 9c1 the result does not depend on the choice of the linear isometry $E \to \mathbb{R}^n$. By translation invariance, the same holds for Euclidean affine spaces.

Riemann integral and Jordan measure are well-defined on every n-dimensional Euclidean affine space, and preserved by affine isometries between these spaces.

9c6 Exercise. Find the volume cut off from the unit ball by the plane $lx + my + nz = p$.

9c7 Exercise. Reformulate the result of 9b8(a) as a geometric theorem about the centroid of an arbitrary right circular cone in a classical Euclidean space.

9d Linear transformation

9d1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case
$$v\big(T(E)\big) = |\det T| v(E) \,.$$

Also, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$|\det T| \int_* f \circ T = \int_* f \quad \text{and} \quad |\det T| \int^* f \circ T = \int^* f \,.$$

Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case

$$|\det T| \int f \circ T = \int f \,.$$

Proof. The Singular Value Decomposition 3i1 gives an orthonormal basis $(a_1, \dots, a_n)$ in $\mathbb{R}^n$ such that vectors $T(a_1), \dots, T(a_n)$ are orthogonal. Invertibility of T ensures that the numbers $s_k = |T(a_k)|$ do not vanish. Taking $b_k = (1/s_k)T(a_k)$ we get an orthonormal basis $(b_1, \dots, b_n)$ such that $T(a_1) = s_1 b_1, \dots, T(a_n) = s_n b_n$.

We have $s_1 \dots s_n = |\det T|$, since the singular values s_k are square roots of the eigenvalues of T^*T (thus, $s_1 \dots s_n = \sqrt{\det(T^*T)} = \sqrt{(\det T)^2} = |\det T|$).

Similarly to the proof of 3c4 we downgrade the two copies of $\mathbb{R}^n$ into a pair of Euclidean vector spaces, choose new bases and upgrade back to two copies of $\mathbb{R}^n$. In other words, we treat $T : \mathbb{R}^n \to \mathbb{R}^n$ as $T : X \to Y$ where X, Y are Euclidean vector spaces, and upgrade X, Y to Cartesian spaces via the bases $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$. The (matrix of the) operator becomes diagonal: $T(x_1, \dots, x_n) = (s_1 x_1, \dots, s_n x_n)$. It remains to apply 6d17 (and 6g12). $\square$

If $|\cdot|_1, |\cdot|_2$ are two Euclidean norms on an n-dimensional vector space, then the ratio of norms $\frac{|\cdot|_1}{|\cdot|_2}$ varies between $\min(s_1, \dots, s_n)$ and $\max(s_1, \dots, s_n)$ (here $s_1, \dots, s_n$ are the singular values), depending on the direction of a vector; but the ratio of volumes $\frac{v_1(\cdot)}{v_2(\cdot)}$ is $s_1 \dots s_n$, invariably.

On an n-dimensional vector or affine space the volume is ill-defined, but Jordan measurability is well-defined, and the ratio $\frac{v(E_1)}{v(E_2)}$ of volumes is well-defined. That is, the volume is well-defined up to a coefficient.

9e Differentiating set functions (again)

Recall Sect. 8c: the derivative F' of a box function F is basically

$$F'(x_0) = \lim_{B \to x_0} \frac{F(B)}{v(B)}$$

(more generally, we use $\liminf$ and $\limsup$ for ${}_*F'$ and ${}^*F'$); the idea of $B \to x_0$ was formalized as

$$B \ni x_0 \quad \text{and} \quad \sup_{x \in B} |x - x_0| < \delta$$

(and ultimately $\delta \to 0+$). Equivalently,

$$\text{(9e1)} \qquad F'(x_0) = a \quad \text{if and only if} \quad \frac{F(B_i)}{v(B_i)} \to a \text{ whenever } B_i \to x_0\,;$$

here "$B_i \to x_0$" means $\forall i\ B_i \ni x_0$ and $\sup_{x \in B_i} |x - x_0| \to 0$ (as $i \to \infty$). More generally, ${}^*F'(x_0)$ is the supremum (in fact, maximum) of $\lim_i \frac{F(B_i)}{v(B_i)}$ over all sequences $(B_i)_i$ such that $B_i \to x_0$ and $\frac{F(B_i)}{v(B_i)}$ converges.

As noted in 8c5, the restriction $B \ni x_0$, unnecessary in Sect. 8,[1] is stipulated in order to conform to the one-dimensional case, and enlarges the class of differentiable box functions. Here (in Sect. 9) we introduce one more restriction (with no one-dimensional counterpart), enlarging further the class of differentiable box functions.

First, we define the *aspect ratio* $\alpha(B)$ of a box B, $\overline{B} = [s_1, t_1] \times \dots \times [s_n, t_n] \subset \mathbb{R}^n$, by

$$\alpha(B) = \frac{\max(t_1 - s_1, \dots, t_n - s_n)}{\min(t_1 - s_1, \dots, t_n - s_n)}\,.$$

Clearly, $\alpha(B) = 1$ if and only if B is a cube.

Second, we redefine the relation $B_i \to x_0$ as follows:

$$\text{(9e2)} \qquad \forall i\ \ B_i \ni x_0\,; \quad \sup_{x \in B_i} |x - x_0| \to 0\,; \quad \sup_i \alpha(B_i) < \infty\,.$$

Third, we redefine accordingly the relation $F'(x_0) = a$ (still (9e1) but with the new interpretation of "$B_i \to x_0$"). Similarly, we redefine ${}_*F'(x_0)$ and ${}^*F'(x_0)$ as (respectively) the infimum and supremum[2] of $\lim_i \frac{F(B_i)}{v(B_i)}$ over all sequences $(B_i)_i$ such that $B_i \to x_0$ (in the new sense) and $\frac{F(B_i)}{v(B_i)}$ converges.

Still, 8c4 holds; $(F + G)' = F' + G'$ etc.

Lemma 8c7 holds, again. The only change needed in the proof is, when taking a partition P_i of a box B_i, to require additionally that $\forall C \in P_i\ \alpha(C) \le \alpha(B_i)$, which is easy to satisfy; and then $\sup_i \alpha(B_i) \le \alpha(B_0) < \infty$.[3] Still, 8c9, (8c10) and 8c11 hold; $F(B) = \int_B F'$, etc.

Why bother about the aspect ratio? The ultimate answer will be given in Sect. 9h, but here is a partial answer. Recall 9c4: $B^\circ = \{x : \|x - x_0\| < 1\}$ and $\overline{B} = \{x : \|x - x_0\| \le 1\}$ for some norm $\|\cdot\|$. As every norm, it is equivalent to the Euclidean norm, $c|\cdot| \le \|\cdot\| \le C|\cdot|$, but these c, C depend on B; if B is small, then C must be large.

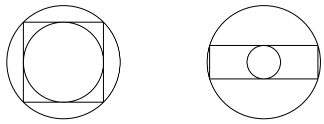
[1] And still unnecessary here in Sect. 9.

[2] Not necessarily maximum.

[3] It is easy to get even more: $\alpha(B_i) \to 1$; we could include such requirement into the definition of "$B_i \to x_0$".

9e3 Lemma. For every box $B \subset \mathbb{R}^n$ there exist $x_0 \in \mathbb{R}^n$, $c \in (0,\infty)$ and a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $B^\circ = \{x : \|x - x_0\| < c\}$, $\overline{B} = \{x : \|x - x_0\| \le c\}$, and

$$\frac{1}{\alpha(B)\sqrt{n}} |\cdot| \le \|\cdot\| \le |\cdot| \,.$$



Proof. We have $\overline{B} = [s_1,t_1]\times\dots\times[s_n,t_n] \subset \mathbb{R}^n$. As noted in the proof of 9c5, $\overline{B} - x_0 = \big\{ x : 2\max\big(\frac{|x_1|}{c_1},\dots,\frac{|x_n|}{c_n}\big) \le 1 \big\}$ where $c_1 = t_1 - s_1, \dots, c_n = t_n - s_n$; note that

$$\alpha(B) = \frac{\max(c_1,\dots,c_n)}{\min(c_1,\dots,c_n)} \,.$$

We take $c = \frac12 \min(c_1,\dots,c_n)$, $C = \max(c_1,\dots,c_n)\sqrt{n}$, and $\|x\| = 2c\max\big(\frac{|x_1|}{c_1},\dots,\frac{|x_n|}{c_n}\big)$, then $\|x\| \le c \iff 2\max\big(\frac{|x_1|}{c_1},\dots,\frac{|x_n|}{c_n}\big) \le 1 \iff x + x_0 \in \overline{B}$, thus $\overline{B} = \{x : \|x - x_0\| \le c\}$; similarly, $B^\circ = \{x : \|x - x_0\| < c\}$.

On one hand, $\forall k \;\; c_k \ge 2c$, therefore $\|x\| \le 2c\max\big(\frac{|x_1|}{2c},\dots,\frac{|x_n|}{2c}\big) = \max(|x_1|,\dots,|x_n|) \le |x|$. On the other hand, $\forall k \;\; c_k \le 2c\alpha(B)$, therefore $\|x\| \ge 2c\max\big(\frac{|x_1|}{2c\alpha(B)},\dots,\frac{|x_n|}{2c\alpha(B)}\big) = \frac{1}{\alpha(B)}\max(|x_1|,\dots,|x_n|) \ge \frac{1}{\alpha(B)\sqrt{n}}|x|$. $\square$

9f Set function induced by mapping

Recall (7d9): $v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x)\,\mathrm{d}x$ for a Jordan set $E \subset \mathbb{R}^{m+n}$ and its sections $E_x = \{y : (x,y) \in E\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$. Denoting $v_n(E_x)$ by $J(x)$ and introducing the projection mapping $\varphi : E \to \mathbb{R}^m$ by $\varphi(x,y) = x$ we have $v_{m+n}\big(\varphi^{-1}(B)\big) = \int_B J$ for all boxes $B \subset \mathbb{R}^m$. According to 7d13 we are interested in more general, nonlinear mappings φ. This leads ultimately to the curvilinear iterated integral treated in Analysis 4. Here are some preliminaries, to be used in the proof of Theorem 9a1.

Consider a mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that the inverse image $\varphi^{-1}(B)$ of every box B is a bounded set. (An example: $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x,y) = x^2 + y^2$.) It leads to a pair of box functions $F_* \le F^*$ (in dimension n),

$$\text{(9f1)} \qquad F_*(B) = v_*(\varphi^{-1}(B^\circ)) \,, \quad F^*(B) = v^*(\varphi^{-1}(\overline{B})) \,,$$

generally not additive but rather superadditive and subadditive: for every partition P of a box B,

$$F_*(B) \ge \sum_{C\in P} F_*(C) \,, \quad F^*(B) \le \sum_{C\in P} F^*(C) \,,$$

which follows from (6g5), (6g6) and the fact that $\varphi^{-1}(C_1^\circ) \cap \varphi^{-1}(C_2^\circ) = \varphi^{-1}(C_1^\circ \cap C_2^\circ) = \emptyset$ when $C_1^\circ \cap C_2^\circ = \emptyset$.

If $F_*(B) = F^*(B)$ then $\varphi^{-1}(B)$ is Jordan measurable, and $\varphi^{-1}(\partial B)$ is of volume zero; if this happens for all B then the box function $F(B) = v(\varphi^{-1}(B))$ is additive. A useful sufficient condition is given below in terms of functions $J_* = {}_*(F_*)'$, $J^* = {}^*(F^*)'$; that is,

$$J_*(x) = \inf_{(B_i)_i} \lim_i \frac{v_*(\varphi^{-1}(B_i^\circ))}{v(B_i)} , \quad J^*(x) = \sup_{(B_i)_i} \lim_i \frac{v^*(\varphi^{-1}(\overline{B}_i))}{v(B_i)} \tag{9f2}$$

where $(B_i)_i$ runs over all sequences of boxes such that $B_i \to x_0$ (see (9e2)) and $\lim_i(\dots)$ exists.

9f3 Proposition. If J_*, J^* are locally integrable and locally equivalent then

$$F_*(B) = F^*(B) = \int_B J_* = \int_B J^*$$

for every box B.

In this case[1]

$$v\big(\varphi^{-1}(B)\big) = \int_B J \tag{9f4}$$

where J is *any* function equivalent to J_*, J^*.

9f5 Exercise. Prove existence of J and calculate it for $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by (a) $\varphi(x, y) = x^2 + y^2$; (b) $\varphi(x, y) = \sqrt{x^2 + y^2}$; (c) $\varphi(x, y) = |x| + |y|$, taking for granted that the area of a disk is πr^2.

9f6 Exercise. Prove existence of J and calculate it for $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\varphi(x, y, z) = \big(\sqrt{x^2 + y^2}, z\big)$, taking for granted Prop. 9b9.

9f7 Exercise. For every partition P of a box B,

$$\min_{C \in P} \frac{F_*(C)}{v(C)} \le \frac{F_*(B)}{v(B)} \le \frac{F^*(B)}{v(B)} \le \max_{C \in P} \frac{F^*(C)}{v(C)} .$$

Prove it.[2]

We generalize 8c9, 8c11.

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[1] Can this happen when $m < n$? If you are intrigued, try the inverse to the mapping of 6c5.

[2] Hint: recall the proof of 8c7.

9f8 Exercise.

$$v(B)\inf_{x\in B} J_*(x) \le F_*(B) \le F^*(B) \le v(B)\sup_{x\in B} J^*(x)\,.$$

Prove it.[1]

9f9 Exercise.

$$\underset{*}{\int}_B J_* \le F_*(B) \le F^*(B) \le \overset{*}{\int}_B J^*\,.$$

Prove it.[2] [3]

Prop. 9f3 follows immediately.

9f10 Remark. Similar statements hold for a mapping defined on a subset of $\mathbb{R}^m$ (rather than the whole $\mathbb{R}^m$). If $\varphi : A \to \mathbb{R}^n$ for a given $A \subset \mathbb{R}^m$ then $\varphi^{-1}(B) \subset A$ for every B, but nothing changes in (9f1), (9f2) and Prop. 9f3. In particular, if $\varphi(A)$ is bounded, then A must be Jordan measurable; otherwise J_*, J^* cannot be integrable and equivalent.

9f11 Example (mind the boundary)**.** Points of $\mathbb{R}^n \setminus \overline{\varphi(A)}$ are irrelevant, but points of $\overline{\varphi(A)} \setminus \varphi(A)$ cannot be ignored, even if they are a set of volume zero. It can happen that $v^*(A) > {}^*\!\int_{\varphi(A)} J^*$.

Recall 8b9: $G = (s_1, t_1) \cup (s_2, t_2) \cup \dots$ is dense in $(0, 1)$, $\sum_k (t_k - s_k) = a < 1$; $v_*(G) = a$, $v^*(G) = 1$. We take $u_k = \sum_{i=k}^{\infty} (t_i - s_i)$, then $u_1 = a$, $u_k \downarrow 0$, and $u_k - u_{k+1} = t_k - s_k$. We define $\varphi : G \to (0, a)$ by

$$\varphi(x) = u_{k+1} + x - s_k \quad \text{for } x \in (s_k, t_k)\,.$$

For every box B such that $\overline{B} \subset (0, a)$ the set $\varphi^{-1}(B)$ is Jordan measurable, and $v\big(\varphi^{-1}(B)\big) = v(B)$ (think, why). Thus, $J_* = J^* = 1$ on $(0, a)$. Nevertheless, $\varphi^{-1}\big((0, a)\big) = G$ fails to be Jordan measurable, and $v^*\big(\varphi^{-1}((0, a))\big) = 1 > a = v\big((0, a)\big)$. Note that $J^*(0) = \infty$, even though $0 \notin \varphi(G)$.

9f12 Remark. If J_*, J^* are integrable and equivalent on a given *closed* box B (and not necessarily on every box) then $v(\varphi^{-1}(C)) = \int_C J$ for every box $C \subset B$.

[1] Hint: similarly to the proof of 8c7; use 9f7.

[2] Similar to 8c11.

[3] Curiously, the left-hand and the right-hand sides differ four times: ${}_*\!\int$, ${}^*\!\int$; $\liminf$, $\limsup$; v_*, v^*; B_i°, $\overline{B}_i$.

9f13 Exercise. Calculate J for the projection mapping $\varphi(x, y) = x$
(a) from the disk $A = \{(x, y) : x^2 + y^2 \le 1\} \subset \mathbb{R}^2$ to $\mathbb{R}$;
(b) from the annulus $A = \{(x, y) : 1 \le x^2 + y^2 \le 4\} \subset \mathbb{R}^2$ to $\mathbb{R}$.
Is J (locally) integrable?

9f14 Exercise. Calculate J for the mapping $\varphi(x) = \sin x$ from the interval $[0, 10\pi] \subset \mathbb{R}$ to $\mathbb{R}$. Is J (locally) integrable?

9g Change of variable in general

9g1 Proposition. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that[1] J_*, J^* are locally integrable and locally equivalent then for every integrable $f : \mathbb{R}^n \to \mathbb{R}$ the function $f \circ \varphi : \mathbb{R}^m \to \mathbb{R}$ is integrable and

$$\int_{\mathbb{R}^m} f \circ \varphi = \int_{\mathbb{R}^n} fJ \, .$$

Proof. First, the claim holds when $f = \mathbb{1}_B$ is the indicator of a box, since

$$\int_{\mathbb{R}^n} fJ = \int_B J \stackrel{\text{(9f4)}}{=} v(\varphi^{-1}(B)) = \int_{\mathbb{R}^m} \mathbb{1}_{\varphi^{-1}(B)} = \int_{\mathbb{R}^m} f \circ \varphi \, .$$

Second, by linearity in f the claim holds whenever f is a step function (on some box, and 0 outside).

Third, given f integrable on a box B (and 0 outside), we consider arbitrary step functions g, h on B such that $g \le f \le h$. We have $g \circ \varphi \le f \circ \varphi \le h \circ \varphi$ and $\int_{\mathbb{R}^m} g \circ \varphi = \int_B gJ$, $\int_{\mathbb{R}^m} h \circ \varphi = \int_B hJ$, thus,

$$\int_B gJ \le {}_*\!\!\int_{\mathbb{R}^m} f \circ \varphi \le {}^*\!\!\int_{\mathbb{R}^m} f \circ \varphi \le \int_B hJ \, ;$$
$$\int_B gJ \le \int_B fJ \le \int_B hJ \, .$$

We take M such that $J(\cdot) \le M$ on B and get

$$\int_B hJ - \int_B gJ = \int_B (h - g)J \le M \int_B (h - g) \, ;$$

thus, integrability of f implies integrability of $f \circ \varphi$ and the needed equality for the integrals. □

[1] We still assume that the inverse image of a box is bounded.

9g2 Corollary. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that J_*, J^* are locally integrable and equivalent then:

(a) for every Jordan measurable set $E \subset \mathbb{R}^n$ the set $\varphi^{-1}(E) \subset \mathbb{R}^m$ is Jordan measurable;

(b) for every integrable $f : E \to \mathbb{R}$ the function $f \circ \varphi$ is integrable on $\varphi^{-1}(E)$, and

$$\int_{\varphi^{-1}(E)} f \circ \varphi = \int_E f J \, .$$

Proof. (a) apply 9g1 to $f = 1\!\!1_E$; (b) apply 9g1 to $f 1\!\!1_E$. □

9g3 Remark. If $\varphi : A \to \mathbb{R}^n$ is such that J_*, J^* are integrable and equivalent on a given *closed* box B (and not necessarily on every box) then for every integrable $f : B \to \mathbb{R}$ the function $f \circ \varphi$ is integrable on the Jordan measurable set $\varphi^{-1}(B)$, and

$$\int_{\varphi^{-1}(B)} f \circ \varphi = \int_B f J \, .$$

Also, 9g2 holds for $E \subset B$.

9g4 Exercise. (a) Prove that $\int_{x^2+y^2 \le 1} f\big(\sqrt{x^2+y^2}\,\big) \, \mathrm{d}x\mathrm{d}y = 2\pi \int_{[0,1]} f(r) r \, \mathrm{d}r$ for every integrable $f : [0,1] \to \mathbb{R}$;

(b) calculate $\int_{x^2+y^2 \le 1} \mathrm{e}^{-(x^2+y^2)/2} \, \mathrm{d}x\mathrm{d}y$. (Could you do it by iterated integrals?)

9h Change of variable for a diffeomorphism

9h1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets and $\varphi : V \to U$ a diffeomorphism, then[1 2]

$$J_*(x) = J^*(x) = |\det(D\psi)_x|$$

for all $x \in U$; here $\psi = \varphi^{-1} : U \to V$.

In the next lemma,[3] given a norm $\|\cdot\|$ on $\mathbb{R}^n$, we denote

$$\mathbf{B}^\circ = \{x : \|x\| < 1\} \, , \quad \overline{\mathbf{B}} = \{x : \|x\| \le 1\} \, ,$$
$$r\mathbf{B}^\circ + x_0 = \{x : \|x - x_0\| < r\} \, , \quad r\overline{\mathbf{B}} + x_0 = \{x : \|x - x_0\| \le r\} \, ;$$

these are Jordan measurable by 9c3.

[1]For J_*, J^* see (9f2).

[2]$\det D\psi$ is called the Jacobian of ψ and often denoted by J_ψ.

[3]We may restrict ourselves to norms such that $\mathbf{B}$ is a box, but this does not simplify the proof of the lemma.

9h2 Lemma. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}^n$ a mapping, $\varepsilon \in (0,1)$, and

$$(1-\varepsilon)\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| \le (1+\varepsilon)\|x_1 - x_2\|$$

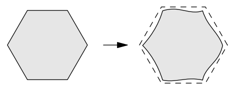
for all $x_1, x_2 \in A$. Then

$$v_*\big(f(r\mathbf{B}^\circ + x)\big) \ge (1-\varepsilon)^n v(r\mathbf{B}^\circ + x)\,,$$
$$v^*\big(f(r\overline{\mathbf{B}} + x)\big) \le (1+\varepsilon)^n v(r\overline{\mathbf{B}} + x)$$

for all x and r such that $r\overline{\mathbf{B}} + x \subset A$ and $(1-\varepsilon)r\mathbf{B}^\circ + f(x) \subset f(A)$.

Proof. The second inequality follows from the inclusion

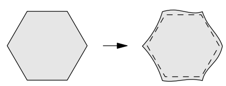
$$f(r\overline{\mathbf{B}} + x) \subset (1+\varepsilon)r\overline{\mathbf{B}} + f(x)\,;$$



proof of this inclusion: for arbitrary $x_1 \in r\overline{\mathbf{B}} + x$, $y_1 = f(x_1)$ is defined, and belongs to $(1+\varepsilon)r\overline{\mathbf{B}} + f(x)$ since $\|y_1 - f(x)\| \le (1+\varepsilon)\|x_1 - x\| \le (1+\varepsilon)r$.

The first inequality follows from the inclusion

$$f(r\mathbf{B}^\circ + x) \supset (1-\varepsilon)r\mathbf{B}^\circ + f(x)\,;$$



proof of this inclusion: for arbitrary $y_1 \in (1-\varepsilon)r\mathbf{B}^\circ + f(x)$ there exists $x_1 \in A$ such that $y_1 = f(x_1)$; and $\|x_1 - x\| \le \frac{1}{1-\varepsilon}\|f(x_1) - f(x)\| = \frac{1}{1-\varepsilon}\|y_1 - f(x)\| < \frac{1}{1-\varepsilon}(1-\varepsilon)r = r$, thus $x_1 \in r\mathbf{B}^\circ + x$ and $y_1 = f(x_1) \in f(r\mathbf{B}^\circ + x)$. $\square$

***Proof** of Prop. 9h1.* Let $x_0 \in U$. Denote $T = (D\psi)_{x_0}$. By Theorem 9d1, $v(T(E)) = |\det T| v(E)$ for every Jordan measurable $E \subset \mathbb{R}^n$. Note that $\varphi^{-1}(E) = \psi(E)$. It is sufficient to prove that

$$\frac{v_*(\psi(B_i^\circ))}{v(T(B_i))} \to 1\,, \quad \frac{v^*(\psi(\overline{B}_i))}{v(T(B_i))} \to 1 \quad \text{whenever } B_i \to x_0\,.$$

Similarly to the proof of 3c4 we downgrade the two copies of $\mathbb{R}^n$ into a pair of affine spaces, and then upgrade them back to $\mathbb{R}^n$ getting $x_0 = 0$, $\psi(x_0) = 0$, and $T = \mathrm{id}$. (In contrast to the proof of 9d1, we do not need Euclidean metric in these affine spaces, since we work with the ratio of volumes of two sets $\psi(B_i), T(B_i)$ in the same space.)

Similarly to the proof of Prop. 3c3 (and 4c7), for every $\varepsilon > 0$ there exists a neighborhood U_ε of 0 such that

$$|(y_1 - y_2) - (x_1 - x_2)| \le \varepsilon |x_1 - x_2|$$

whenever $x_1, x_2 \in U_\varepsilon$ and $y_1 = \psi(x_1)$, $y_2 = \psi(x_2)$.

The set $V_\varepsilon = \psi(U_\varepsilon)$ is a neighborhood of 0 (since ψ is open, recall 3b6). Given $B_i \to x_0$, we take M such that $\forall i \;\; \alpha(B_i) \le M/\sqrt{n}$; using Lemma 9e3 we get points $x_i \to 0$, numbers $c_i \to 0+$ and norms $\|\cdot\|_i$ such that $\overline{B}_i = \{x : \|x - x_i\|_i \le c_i\} = c_i \overline{\mathbf{B}}_i + x_i$ and

$$\frac{1}{M}|\cdot| \le \|\cdot\|_i \le |\cdot| \, .$$

For all $x_1, x_2 \in U_\varepsilon$, denoting $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$, we have

$$\|(y_1 - y_2) - (x_1 - x_2)\|_i \le |(y_1 - y_2) - (x_1 - x_2)| \le \varepsilon |x_1 - x_2| \le M\varepsilon \|x_1 - x_2\| \, ,$$

therefore (assuming $\varepsilon < 1/M$),

$$(1 - M\varepsilon)\|x_1 - x_2\|_i \le \|y_1 - y_2\|_i \le (1 + M\varepsilon)\|x_1 - x_2\|_i \, .$$

Claim: For all i large enough, $c_i \overline{\mathbf{B}}_i + x_i \subset U_\varepsilon$ and $c_i \mathbf{B}_i^\circ + y_i \subset V_\varepsilon$, where $y_i = f(x_i)$. Proof of the claim. First, all x such that $\|x - x_i\|_i \le c_i$ satisfy $|x| \le |x - x_i| + |x_i| \le M\|x - x_i\|_i + |x_i| \le Mc_i + |x_i| \to 0$ as $i \to \infty$. Second, all y such that $\|y - y_i\|_i \le c_i$ satisfy $|y| \le |y - y_i| + |y_i| \le M\|y - y_i\|_i + |y_i| < Mc_i + |y_i| \to 0$ as $i \to \infty$.

Applying Lemma 9h2 to $\|\cdot\| = \|\cdot\|_i$, $A = U_\varepsilon$, $f = \psi$, $M\varepsilon$, $x = x_i$ and $r = c_i$ we get, for all i large enough,

$$\begin{aligned} v_*\big(\psi(B_i^\circ)\big) &\ge (1 - M\varepsilon)^n v(B_i) \, , \\ v^*\big(\psi(\overline{B}_i)\big) &\le (1 + M\varepsilon)^n v(B_i) \, , \end{aligned}$$

that is,

$$(1 - M\varepsilon)^n \le \frac{v_*(\psi(B_i^\circ))}{v(T(B_i))} \le \frac{v^*(\psi(\overline{B}_i))}{v(T(B_i))} \le (1 + M\varepsilon)^n \, .$$

$\square$

According to 9f11 we should examine J^* on the boundary of U. But this is hard. A diffeomorphism $V \to U$ need not extend to a homeomorphism $\overline{V} \to \overline{U}$. Here are two simple counterexamples:[1]

(a) $V = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$, $U = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$, $\varphi(x) = x + \frac{x}{|x|}$;

(b) $V = U = \{x \in \mathbb{R}^2 : |x| < 1\}$, $\varphi(r\cos\theta, r\sin\theta) = \big(r\cos(\theta + \frac{1}{1-r}), r\sin(\theta + \frac{1}{1-r})\big)$.

Fortunately, we have another way: approximation from inside (since, in contrast to 9f11, we *assume* that U, V are Jordan measurable).

[1] Far not the worst case.

***Proof** of Theorem 9a1.* We prove the equivalent formulation given by 9a2: $\varphi : V \to U$ a diffeomorphism, $f : U \to \mathbb{R}$ bounded on U, and $f|\det D\psi|$ bounded on U, where $\psi = \varphi^{-1} : U \to V$. We have to prove that $\int_V f \circ \varphi = \int_U f|\det D\psi|$, provided that one integrand is integrable.

By 9h1, J_* and J^* are integrable and equivalent (moreover, continuous and equal) on every box B such that $\overline{B} \subset U$. By 9g3, $\psi(B) \subset V$ is Jordan measurable, and

$$\int_{\psi(B)} f \circ \varphi = \int_B fJ$$

whenever f is integrable on B. In particular, taking $f = 1\!\!1_E$, we get $v\big(\psi(E)\big) = \int_E J$ for all Jordan measurable $E \subset B$. Thus,

if $E \subset B$ is of volume zero, then $\psi(E)$ is also of volume zero.

Similarly to the proof of 8e6 we take a box $B \subset \mathbb{R}^n$ such that $U \subset B^\circ$, and for arbitrary partition P of B we consider

$$K_P = \bigcup_{C \in P, \overline{C} \subset U} \overline{C}\,.$$

Taking into account that the boundaries ∂C do not matter we get $\int_{\psi(K_P)} f \circ \varphi = \int_{K_P} fJ$ whenever f is integrable on K_P. And if $E \subset K_P$ is of volume zero, then $\psi(E)$ is of volume zero.

We take a sequence of partitions $P_1, P_2, \dots$ of B such that each P_{i+1} is a refinement of P_i, and

$$\forall C \in P_i \;\; \operatorname{diam}(C) \le \varepsilon_i\,, \quad \varepsilon_i \to 0\,.$$

Claim: the corresponding sets $K_i = K_{P_i}$ satisfy $K_i^\circ \uparrow U$. Proof of the claim: clearly, $K_i \uparrow$. Given $x \in U$, we take i such that $\varepsilon_i < \operatorname{dist}(x, \mathbb{R}^n \setminus U)$, and $C \in P_i$ such that $x \in \overline{C}$, then $\overline{C} \subset U$ (since $\operatorname{diam}(C) \le \varepsilon_i$), thus $x \in \overline{C} \subset K_i$; and moreover, $x \in K_i^\circ$, since $\operatorname{dist}(y, \mathbb{R}^n \setminus U) > \varepsilon_i$ for all y near x.

If f has a compact support inside U and fJ is integrable, then f is integrable (since the continuous function $1/J$ is bounded on the support of f), therefore $f \circ \varphi$ is integrable[1] and $\int_V f \circ \varphi = \int_U fJ$ (since the support of f is contained in some K_i°).

The same holds for the inverse diffeomorphism; if an integrable g has a compact support inside V, then $g \circ \psi$ is integrable. Taking $g = f \circ \varphi$ we see that integrability of $f \circ \varphi$ implies integrability of f (and fJ). Thus,

$$(f \circ \varphi \text{ is integrable}) \quad \Longleftrightarrow \quad (fJ \text{ is integrable})\,,$$

$$\text{and in this case } \int_V f \circ \varphi = \int_U fJ$$

[1]Alternatively you may prove the integrability via 8f1.

whenever f has a compact support inside U. We have to get rid of this assumption.

The relation $K_i^\circ \uparrow U$ implies $\psi(K_i^\circ) \uparrow V$. By 8e9, $v_*(K_i^\circ) \uparrow v_*(U)$ and $v_*\big(\psi(K_i^\circ)\big) \uparrow v_*(V)$. Taking into account Jordan measurability of these sets we get

$$v(U \setminus K_i^\circ) \to 0\,, \quad v(V \setminus \psi(K_i^\circ)) \to 0 \qquad \text{as } i \to \infty\,.$$

Using boundedness of f (and therefore, $f \circ \varphi$) and fJ, we get integral convergence (recall Sect. 6e):

$$(f \circ \varphi) \cdot 1\!\!1_{\psi(K_i^\circ)} \to f \circ \varphi\,, \quad fJ \cdot 1\!\!1_{K_i^\circ} \to fJ$$

as $i \to \infty$. It remains to use 6e3(b). $\square$

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