7 Linear change of variables

7a Admissible sets in vector spaces

7a1 Proposition. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every $E \subset \mathbb{R}^n$,

$$A(E) \text{ is admissible} \quad \Longleftrightarrow \quad E \text{ is admissible.}$$

7a2 Lemma. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then, for every bounded set $Z \subset \mathbb{R}^n$ of volume 0, the set $A(Z)$ has volume 0.

Proof. The image $A(Q)$ of the cube $Q = [0,1]^n$ is bounded (think, why). We take a box B such that $A(Q) \subset B$ and get $v^*\big(A(Q)\big) \le v(B) < \infty$. Moreover, using 4h2 and (4h6) we get[1] for all N and $k \in \mathbb{Z}^N$

$$v^*\big(A(2^{-N}(Q+k))\big) \le M \cdot 2^{-nN}$$

where $M = v(B)$.

Using subadditivity of the outer volume,[2]

$$v^*(E \cup F) \le v^*(E) + v^*(F)\,, \tag{7a3}$$

we get for arbitrary bounded E,

$$v^*\big(A(E)\big) \le \sum_{k:2^{-N}(Q+k)\cap E\ne\emptyset} v^*\big(A(2^{-N}(Q+k))\big) \le M \cdot 2^{-nN} \sum_{k:2^{-N}(Q+k)\cap E\ne\emptyset} 1 = MU_N(\mathbb{1}_E)\,;$$

for $N \to \infty$ it gives $v^*\big(A(E)\big) \le Mv^*(E)$. Thus, $v^*(Z) = 0$ implies $v^*(A(Z)) = 0$. □

7a4 Remark. Let A be invertible. Then Z has volume 0 if and only if $A(Z)$ has volume 0.

[1] Since $A(2^{-N}(Q+k)) \subset 2^{-N}(B + A(k))$.

[2] Indeed, ${}^*\!\!\int \mathbb{1}_{E\cup F} \le {}^*\!\!\int (\mathbb{1}_E + \mathbb{1}_F) \le {}^*\!\!\int \mathbb{1}_E + {}^*\!\!\int \mathbb{1}_F$ by 4c7.

***Proof** of Prop. 7a1.* We'll prove that $A(E)$ is admissible whenever E is admissible (then, applying it to A^{-1}, we get the converse implication). By 6b8(b), ∂E has volume 0; by 7a2, $A(\partial E)$ has volume 0; also, $A(\partial E) = \partial A(E)$, since A is a homeomorphism; thus, $\partial A(E)$ has volume 0; by 6b8(b) (again), $A(E)$ is admissible. □

Similarly to Sect. 1f we conclude.

The notion "admissible set" is insensitive to a change of basis. This notion is well-defined in every n-dimensional vector space, and preserved by isomorphisms of these spaces.

The same holds for the notion "volume 0".

7a5 Exercise. (a) Every bounded subset of a vector subspace $V_1 \subsetneqq V$ has volume 0;

(b) every vector subspace $V_1 \subsetneqq V$ has measure 0.
Prove it.[1]

7b Volume in Euclidean spaces

7b1 Proposition. If a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Euclidean metric (that is, $|Ax| = |x|$ for all $x \in \mathbb{R}^n$), then it preserves volume (that is, $v(A(E)) = v(E)$ for all admissible $E \subset \mathbb{R}^n$).

Rotation invariance of volume, at last!

7b2 Lemma. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then there exists $C \in (0, \infty)$ such that, for every admissible $E \subset \mathbb{R}^n$,

$$v\big(A(E)\big) = Cv(E)\,.$$

Proof. We take $C = v(A(Q))$ where $Q = [0, 1]^n$ (admissibility of $A(Q)$ being ensured by 7a1, and $C \neq 0$ by 7a4). Similarly to the proof of 7a2, using 4h2 and (4h6) we get for all N and $k \in \mathbb{Z}^N$

$$v\big(A(2^{-N}(Q + k))\big) = C \cdot 2^{-nN}\,.$$

For $k \neq \ell$ the set $A(2^{-N}(Q + k)) \cap A(2^{-N}(Q + \ell)) = A\big((2^{-N}(Q + k)) \cap (2^{-N}(Q + \ell))\big)$ has volume 0 by 7a2. Using additivity of volume 4d3, we get

$$v\big(A(E)\big) \le \sum_{k:2^{-N}(Q+k)\cap E\neq\emptyset} v\big(A(2^{-N}(Q + k))\big) = CU_N(\mathbb{1}_E)\,,$$

[1] Hint: change of basis.

and similarly,

$$v\big(A(E)\big) \ge \sum_{k:2^{-N}(Q+k)\subset E} v\big(A(2^{-N}(Q+k))\big) = CL_N(\mathbb{1}_E)\,;$$

for $N \to \infty$ it gives $v\big(A(E)\big) = Cv(E)$. □

If A is of the form $A(x_1,\dots,x_n) = (a_1x_1,\dots,a_nx_n)$ (that is, diagonal matrix), then $C = |a_1 \dots a_n|$ by 4h4.

***Proof** of Prop. 7b1.* The constant C given by 7b2 is equal to 1, since the ball $E = \{x : |x| \le 1\}$ (admissible by 4i4, and of non-zero volume since $E^\circ \ne \emptyset$) satisfies $A(E) = E$. □

> Volume is insensitive to a change of orthonormal basis.
> It is well-defined in every n-dimensional Euclidean space, and preserved by isomorphisms of these spaces.

Now we are in position to find the constant C for arbitrary A.

7b3 Theorem. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every admissible $E \subset \mathbb{R}^n$,

$$v\big(A(E)\big) = |\det A|\, v(E)\,.$$

Recall the singular value decomposition (Sect. 3d).[1]

Proof. By 3d2, some change of two orthonormal bases in $\mathbb{R}^n$ turns A into a diagonal matrix whose diagonal elements are the singular values $s_1,\dots,s_n$ of A. The constant C, insensitive to this change of bases, is equal to $s_1 \dots s_n$ by 4h4. It remains to prove an algebraic fact: $|\det A| = s_1 \dots s_n$.

A change of orthonormal bases multiplies a matrix from the left and from the right by orthogonal matrices; it means, a matrix U such that $|Ux| = |x|$ for all x. It follows that $\langle x,y\rangle = \langle Ux,Uy\rangle = \langle U^*Ux,y\rangle$, thus $\mathrm{id} = U^*U$; $1 = \det(U^*U) = \det(U^*)\det U = (\det U)^2$; $\det U = \pm 1$. □

If $|\cdot|_1, |\cdot|_2$ are two Euclidean norms on an n-dimensional vector space, then the ratio of norms $\frac{|\cdot|_1}{|\cdot|_2}$ varies between $\min(s_1,\dots,s_n)$ and $\max(s_1,\dots,s_n)$ (here $s_1,\dots,s_n$ are the singular values), depending on the direction of a vector; but the ratio of volumes $\frac{v_1(\cdot)}{v_2(\cdot)}$ is $s_1 \dots s_n$, invariably.

[1]Some linear algebra is needed here. Many authors decompose an arbitrary matrix into the product of elementary matrices (of three types). But I prefer the singular value decomposition.

On an n-dimensional vector space the volume is ill-defined, but admissibility is well-defined, and the ratio $\frac{v(E_1)}{v(E_2)}$ of volumes is well-defined. That is, the volume is well-defined up to a coefficient.

7b4 Exercise. Find the volume cut off from the unit ball by the plane $ax + by + cz = t$.

7b5 Exercise. Let vectors $h_1, \dots, h_n \in \mathbb{R}^n$ be linearly independent, and $C = |\det(h_1, \dots, h_n)|$.

(a) The parallelotope $E = \{u_1 h_1 + \dots + u_n h_n : 0 \le u_1, \dots, u_n \le 1\}$ is admissible, and $v(E) = C$.

(b) The simplex $E = \{u_1 h_1 + \dots + u_n h_n : u_1, \dots, u_n \ge 0, u_1 + \dots + u_n \le 1\}$ is admissible, and $v(E) = \frac{1}{n!} C$.

(c) The ellipsoid $E = \{u_1 h_1 + \dots + u_n h_n : u_1^2 + \dots + u_n^2 \le 1\}$ is admissible, and $\frac{1}{C} v(E)$ is equal to the volume of the n-dimensional unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$.

Prove it.[1]

7c Linear change of variables

7c1 Theorem. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$|\det A| \int_* f \circ A = \int_* f \quad \text{and} \quad |\det A| \int^* f \circ A = \int^* f \, .$$

Thus, $f \circ A$ is integrable if and only if f is integrable, and in this case

$$|\det A| \int f \circ A = \int f \, .$$

Proof. First, consider the indicator $f = 1\!\!1_E$ of an admissible set $E \subset \mathbb{R}^n$. We have $f \circ A = 1\!\!1_{A^{-1}(E)}$ (think, why); this function is integrable by 7a1, and

$$\int f \circ A = v(A^{-1}(E)) = |\det A^{-1}| v(E) = \frac{1}{|\det A|} \int f$$

by 7b3.

In particular, it holds for indicators of boxes. Taking linear combinations we see that the equality $|\det A| \int f \circ A = \int f$ holds for all step functions f.

[1] Hint: (b) use 5e17.

Now, the general case. Given $\varepsilon > 0$, 4g6 gives a step function $h \ge f$ such that $\int h \le \varepsilon + {}^*\!\!\int f$. We have

$$|\det A| \,{}^*\!\!\int f \circ A \le |\det A| \int h \circ A = \int h \le \varepsilon + {}^*\!\!\int f$$

for all $\varepsilon > 0$, thus,

$$|\det A| \,{}^*\!\!\int f \circ A \le {}^*\!\!\int f \,.$$

Applying it to A^{-1} and $f \circ A$ we get $|\det A^{-1}| {}^*\!\!\int f \circ A \circ A^{-1} \le {}^*\!\!\int f \circ A$, that is, ${}^*\!\!\int f \le |\det A| {}^*\!\!\int f \circ A$. Thus, $|\det A| {}^*\!\!\int f \circ A = {}^*\!\!\int f$. Similarly (or using $(-f)$), $|\det A| \,{}_*\!\!\int f \circ A = {}_*\!\!\int f$. □

In the exercises below you may start with changing basis, or with opening brackets. When really needed, use iterated integral (and scaling). Sometimes 5e10(c) may help. Think, which way is shorter. A hint: in order to prove that an integral is equal to 0 it is sufficient to find a change of basis that flips the sign of the integral.

7c2 Exercise. If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then

$$\iiint\limits_{x^2+y^2+z^2<1} (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0\,.$$

Prove it.

7c3 Exercise. Find the mean value of the function $(x, y, z) \mapsto (ax+by+cz)^2$ on the ball $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$.[1]

7c4 Exercise. Find the mean value of the function $(x, y, z) \mapsto (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)$ on the ball $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$.[2]

7c5 Exercise. Let $h_1, h_2, h_3 \in \mathbb{R}^3$ and $t_1, t_2, t_3 \in \mathbb{R}$. Find the mean value of the function $x \mapsto (\langle h_1, x\rangle + t_1)(\langle h_2, x\rangle + t_2)(\langle h_3, x\rangle + t_3)$ on the ball $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$.[3]

[1]Answer: $\frac{1}{5}(a^2 + b^2 + c^2)$.
[2]Answer: $\frac{1}{5}(a_1a_2 + b_1b_2 + c_1c_2)$.
[3]Answer: $\frac{1}{5}(\langle h_1, h_2\rangle t_3 + \langle h_1, h_3\rangle t_2 + \langle h_2, h_3\rangle t_1) + t_1t_2t_3$.