A Monotone convergence for Riemann integral

This appendix is not a part of the course. Read it if you are interested in the proof of Theorem 9e5.

First, two quotes.

> *The Dominated Convergence Theorem is a fundamental result in Real Analysis, often presented as one of the main features of Lebesgue integral. Due to the omnipresence of Lebesgue integral in real analysis one might think that nothing of this kind works in the context of Riemann integral. This is not true because the discovery by C. Arzelà [2] of the Bounded Convergence Theorem, preceded by more than a decade the famous work of Lebesgue. [...]*
> *Theorem 1 (The Bounded Convergence Theorem). [...]*
> *Theorem 2 (The Monotone Convergence Theorem). [...]* [1]

> *Despite the availability of this variety of elementary proofs for Arzela's theorem, the present author finds that in most textbooks on analysis, whose authors have chosen to treat the Riemann integral rather than the Lebesgue integral, Arzela's theorem is not mentioned, or, if it is mentioned, it is rarely accompanied by a correct proof or by any proof at all.* [2]

Here is Theorem 9e5 formulated again.

A1 Theorem (monotone convergence for Riemann integral)**.** [3] If $f, f_1, f_2, \dots : \mathbb{R}^n \to \mathbb{R}$ are integrable and $f_k \uparrow f$,[4] then $\int_{\mathbb{R}^n} f_k \uparrow \int_{\mathbb{R}^n} f$.

A2 Lemma. If continuous functions $h, h_1, h_2, \dots : \mathbb{R}^n \to \mathbb{R}$ with bounded supports satisfy $h_k \uparrow h$, then $\int_{\mathbb{R}^n} h_k \uparrow \int_{\mathbb{R}^n} h$.

Proof. WLOG, $h_1 \ge 0$ (otherwise use $h_k - h_1 \uparrow h - h_1$). Open sets[5]

$$E_k = \{(x,t) : 0 < t < h_k(x)\}\,, \quad E = \{(x,t) : 0 < t < h(x)\} \quad \subset \mathbb{R}^{n+1}$$

are admissible and satisfy $v(E_k) = \int h_k$, $v(E) = \int h$ by 4i1. Clearly, $E_k \uparrow E$; by 9b11, $v(E_k) \uparrow v(E)$. □

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[1] C.P. Niculescu, F. Popovici "The Monotone Convergence Theorem for the Riemann integral", Ann. Univ. Craiova **38**:2 (2011) 55–58.

[2] W.A.J. Luxemburg "Arzelà's Dominated Convergence Theorem for the Riemann integral", Amer. Math. Monthly **78**:9 (1971) 970–979.

[3] The *proper* Riemann integral is meant. Thus, each integrable function is bounded, with bounded support.

[4] Pointwise convergence (monotone).

[5] Continuity of the functions ensures that the sets are open.

A3 Lemma. Let $\varepsilon > 0$, and integrable $f_1, f_2, \dots : \mathbb{R}^n \to \mathbb{R}$ satisfy $f_1 \le f_2 \le \dots$ Then there exist continuous functions $h_1, h_2, \dots : \mathbb{R}^n \to \mathbb{R}$ with bounded supports such that $h_1 \le h_2 \le \dots$, $f_k \le h_k$ and $\int_{\mathbb{R}^n} h_k \le \int_{\mathbb{R}^n} f_k + \varepsilon$ for all k.

Proof. We take $\varepsilon_1, \varepsilon_2, \dots > 0$ such that $\sum_k \varepsilon_k \le \varepsilon$. For each k we take[1] continuous $g_k : \mathbb{R}^n \to \mathbb{R}$ with bounded support such that $f_k \le g_k$ and $\int g_k \le \int f_k + \varepsilon_k$. We define $h_k = \max(g_1, \dots, g_k)$. Clearly, $h_1 \le h_2 \le \dots$ and $f_k \le h_k$. It remains to prove that $\int h_k \le \int f_k + \varepsilon$.

We have

$$\begin{aligned} h_2 - f_2 = \max(g_1, g_2) - f_2 = \max(g_1 - f_2, g_2 - f_2) \le \\ \le \max(g_1 - f_1, g_2 - f_2) \le (g_1 - f_1) + (g_2 - f_2) \, , \end{aligned}$$

thus, $\int h_2 - \int f_2 \le \int (g_1 - f_1) + \int (g_2 - f_2) \le \varepsilon_1 + \varepsilon_2$. Similarly, $\int h_k - \int f_k \le \varepsilon_1 + \dots + \varepsilon_k \le \varepsilon$ for all k. □

***Proof** of Th. A1.* Clearly, $\int f_k \uparrow a \le \int f$; we have to prove that $\int f \le a$, that is, $\int g \le a$ whenever continuous $g : \mathbb{R}^n \to \mathbb{R}$ with bounded support satisfies $g \le f$. Given $\varepsilon > 0$, we take h_k according to A3, consider functions $\min(h_k, g)$ and note that $\min(h_k, g) \uparrow g$ (since $\lim_k \min(h_k, g) \ge \lim_k \min(f_k, g) = \min(f, g) = g$). By A2, $\int \min(h_k, g) \uparrow \int g$. On the other hand, $\int \min(h_k, g) \le \int h_k \le \int f_k + \varepsilon \uparrow a + \varepsilon$, thus $\int g \le a + \varepsilon$ for all ε. □

See also the cited articles (and references therein) for more on convergence theorems for Riemann integral.

[1] By 4g9(c).