4 Divergence theorem and its consequences

The divergence theorem sheds light on harmonic functions and differential forms.

4a Divergence and flux

We return to the case treated before, in the end of Sect. 3b: $G \subset \mathbb{R}^N$ is a smooth set. Recall the outward unit normal vector $\mathbf{n}_x$ for $x \in \partial G$.

4a1 Definition. For a continuous $F : \partial G \to \mathbb{R}^N$, the (outward) *flux* of (the vector field) F through ∂G is

$$\int_{\partial G} \langle F, \mathbf{n} \rangle \, .$$

(The integral is interpreted according to (2d8).)

If a vector field F on $\mathbb{R}^3$ is the velocity field of a fluid, then the flux of F through a surface is the amount[1] of fluid flowing through the surface (per unit time).[2] If the fluid is flowing parallel to the surface then, evidently, the flux is zero.

We continue similarly to Sect. 3b. Let $F \in C^1(G \to \mathbb{R}^N)$, with DF bounded (on G). Recall that, by 3b6, boundedness of DF on G ensures that F extends to $\overline{G}$ by continuity (and therefore is bounded). *In such cases we always use this extension.* The mapping $\tilde F : \mathbb{R}^N \setminus \partial G \to \mathbb{R}^N$ defined by

$$\tilde F(x) = \begin{cases} F(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin \overline{G} \end{cases}$$

[1] The volume is meant, not the mass. However, these are proportional if the density (kg/m^3) of the matter is constant (which often holds for fluids).

[2] See also mathinsight.

is continuous up to ∂G, and

$$\tilde{F}(x - 0\mathbf{n}_x) = F(x) \,, \quad \tilde{F}(x + 0\mathbf{n}_x) = 0 \,;$$
$$\operatorname{div}_{\text{sng}} \tilde{F}(x) = -\langle F(x), \mathbf{n}_x \rangle \,.$$

By Theorem 3e3 (applied to $\tilde{F}$ and $K = \partial G$),

$$\int_G \operatorname{div} F = \int_{\partial G} \langle F, \mathbf{n} \rangle \,, \tag{4a2}$$

just the flux. The divergence theorem, formulated below, is thus proved.[1]

4a3 Theorem (*Divergence theorem*)**.** Let $G \subset \mathbb{R}^N$ be a smooth set, $F \in C^1(G \to \mathbb{R}^N)$, with DF bounded on G. Then the integral of $\operatorname{div} F$ over G is equal to the (outward) flux of F through ∂G.

In particular, if $\operatorname{div} F = 0$, then $\int_{\partial G} \langle F, \mathbf{n} \rangle = 0$.

4a4 Exercise. $\operatorname{div}(fF) = f \operatorname{div} F + \langle \nabla f, F \rangle$ whenever $f \in C^1(G)$ and $F \in C^1(G \to \mathbb{R}^N)$

Prove it.

Thus, the divergence theorem, applied to fF when $f \in C^1(G)$ with bounded ∇f, and $F \in C^1(G \to \mathbb{R}^N)$ with bounded DF, gives a kind of integration by parts, similar to (3b12):

$$\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, \mathbf{n} \rangle - \int_G f \operatorname{div} F \,. \tag{4a5}$$

In particular, if $\operatorname{div} F = 0$, then $\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, \mathbf{n} \rangle$

Here is a useful special case. We mean by a radial function a function of the form $f : x \mapsto g(|x|)$ where $g \in C^1(0, \infty)$, and by a radial vector field $F : x \mapsto g(|x|)x$. Clearly, $f \in C^1(\mathbb{R}^N \setminus \{0\})$ and $F \in C^1(\mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N)$.

4a6 Exercise. (a) If $f(x) = g(|x|)$, then $\nabla f(x) = \frac{g'(|x|)}{|x|} x$;

(b) if $F(x) = g(|x|)x$, then $\operatorname{div} F(x) = |x| g'(|x|) + N g(|x|)$;

(c) if $F(x) = g(|x|)x$, then the (outward) flux of F through the boundary of the ball $\{x : |x| < r\}$ is $c r^N g(r)$, where $c = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of the unit sphere.

Prove it.[2]

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[1] Divergence is often explained in terms of sources and sinks (of a moving matter). But be careful; the flux of a velocity field is the amount (per unit time) as long as "amount" means "volume". If by "amount" you mean "mass", then you need the vector field of momentum, not velocity; multiply the velocity by the density of the matter. However, the problem disappears if the density is constant (which often holds for fluids).

[2] Hint: (b) use (a) and 4a4.

Taking $G = \{x : a < |x| < b\}$ and $F(x) = g(|x|)x$, we see that $\int_G \operatorname{div} F = \int_a^b cr^{N-1}\big(rg'(r)+Ng(r)\big)\,\mathrm{d}r$ by 4a6(b) and (generalized) 3c8; and on the other hand, $\int_{\partial G}\langle F, \mathbf{n}\rangle = cr^N g(r)|_{r=a}^b$ by 4a6(c). Well, $\frac{\mathrm{d}}{\mathrm{d}r}\big(r^N g(r)\big) = r^{N-1}\big(rg'(r) + Ng(r)\big)$, as it should be according to (4a2).

Zero gradient is trivial, but zero divergence is not. For a radial vector field, zero divergence implies that $r^N g(r)$ does not depend on r, that is, $g(r) = \frac{\text{const}}{r^N}$ (and indeed, in this case $rg'(r) + Ng(r) = 0$);

(4a7)
$$F(x) = \frac{\text{const}}{|x|^N}x\,; \quad \operatorname{div} F(x) = 0 \quad \text{for } x \ne 0\,;$$
$$\int_{\partial G}\langle F, \mathbf{n}\rangle = 0 \quad \text{when } \overline{G} \not\ni 0\,;$$

note that the latter equality fails for a ball. The flux through a sphere is

(4a8)
$$\int_{|x|=r}\langle F, \mathbf{n}\rangle = \text{const}\cdot\int_{|x|=1} 1 = \text{const}\cdot\frac{2\pi^{N/2}}{\Gamma(N/2)}$$

where 'const' is as in (4a7). The same holds for arbitrary smooth set $G \ni 0$:

(4a9)
$$\int_{\partial G}\langle F, \mathbf{n}\rangle = \text{const}\cdot\frac{2\pi^{N/2}}{\Gamma(N/2)}\,.$$

Proof: we take $\varepsilon > 0$ such that $\{x : |x| \le \varepsilon\} \subset G$; the set $G_\varepsilon = \{x \in G : |x| > \varepsilon\}$ is smooth; by (4a7), $\int_{\partial G_\varepsilon}\langle F, \mathbf{n}\rangle = 0$; and $\partial G_\varepsilon = \partial G \uplus \{x : |x| = \varepsilon\}$.

4b Piecewise smooth case

We want to apply the divergence theorem 4a3 to the open cube $G = (0,1)^N$, but for now we cannot, since the boundary ∂G is not a manifold. Rather, ∂G consists of $2N$ disjoint cubes of dimension $n = N-1$ ("hyperfaces") and a finite number[1] of cubes of dimensions $0, 1, \dots, n-1$. $n=N-1$

For example, $\{1\}\times(0,1)^n$ is a hyperface.

Each hyperface is an n-manifold, and has exactly two orientations. Also, the outward unit normal vector $\mathbf{n}_x$ is well-defined at every point x of a hyperface.

For example, $\mathbf{n}_x = e_1$ for every $x \in \{1\}\times(0,1)^n$.

For a function f on ∂G we define $\int_{\partial G} f$ as the sum of integrals over the $2N$ hyperfaces; that is,

(4b1)
$$\int_{\partial G} f = \sum_{i=1}^N \sum_{x_i=0,1} \int\!\cdots\!\int_{(0,1)^n} f(x_1,\dots,x_N)\prod_{j:j\ne i}\mathrm{d}x_j\,,$$

[1] In fact, $3^N - 1 - 2N$.

provided that these integrals are well-defined, of course.

For a vector field $F \in C(\partial G \to \mathbb{R}^N)$ we define the flux of F through ∂G as $\int_{\partial G} \langle F, \mathbf{n} \rangle$. Note that

$$\text{(4b2)} \qquad \int_{\partial G} \langle F, \mathbf{n} \rangle = \sum_{i=1}^{N} \sum_{x_i=0,1} (2x_i - 1) \underset{(0,1)^n}{\int \cdots \int} F_i(x_1, \dots, x_N) \prod_{j:j \ne i} \mathrm{d}x_j \, .$$

It is surprisingly easy to prove the divergence theorem for the cube. (Just from scratch; no need to use 4a3, nor 3e3.)

4b3 Proposition (divergence theorem for cube)**.** Let $F \in C^1\big((0,1)^N \to \mathbb{R}^N\big)$, with DF bounded. Then the integral of $\operatorname{div} F$ over $(0,1)^N$ is equal to the (outward) flux of F through the boundary.

(As before, boundedness of DF ensures that F extends to $[0,1]^N$ by continuity; recall 3b6.)

Proof.

$$\int_0^1 D_1 F_1(x_1, \dots, x_N) \, \mathrm{d}x_1 = F_1(1, x_2, \dots, x_N) - F_1(0, x_2, \dots, x_N) = \\ = \sum_{x_1=0,1} (2x_1 - 1) F_1(x_1, \dots, x_N) \, ;$$

$$\underset{(0,1)^N}{\int \cdots \int} D_1 F_1 = \sum_{x_1=0,1} (2x_1 - 1) \underset{(0,1)^n}{\int \cdots \int} F_1(x_1, \dots, x_N) \, \mathrm{d}x_2 \dots \mathrm{d}x_N \, ;$$

similarly, for each $i = 1, \dots, N$,

$$\underset{(0,1)^N}{\int \cdots \int} D_i F_i = \sum_{x_i=0,1} (2x_i - 1) \underset{(0,1)^n}{\int \cdots \int} F_i \prod_{j:j \ne i} \mathrm{d}x_j \, ;$$

it remains to sum over i. □

The same holds for every box, of course.

A box is only one example of a bounded regular open set $G \subset \mathbb{R}^N$ such that ∂G is not an n-manifold and still, the divergence theorem holds as $\int_G \operatorname{div} F = \int_{\partial G \setminus Z} \langle F, \mathbf{n} \rangle$ for some closed set $Z \subset \partial G$ such that $\partial G \setminus Z$ is an n-manifold of finite n-dimensional volume. For the cube (or box), $\partial G \setminus Z$ is the union of the $2N$ hyperfaces, and Z is the union of cubes (or boxes) of smaller (than $N-1$) dimensions.

4b4 Definition. We say[1] that the divergence theorem holds for G and $\partial G \setminus Z$, if

$G \subset \mathbb{R}^N$ is a bounded regular open set,

$Z \subset \partial G$ is a closed set,

$\partial G \setminus Z$ is an n-manifold of finite n-dimensional volume, and

$\int_G \operatorname{div} F = \int_{\partial G \setminus Z} \langle F, \mathbf{n} \rangle$ for all $F \in C(\overline{G} \to \mathbb{R}^N)$ such that $F|_G \in C^1(G \to \mathbb{R}^N)$ and DF is bounded on G.

4b5 Exercise (PRODUCT)**.** Let $G_1 \subset \mathbb{R}^{N_1}$, $Z_1 \subset \partial G_1$, and $G_2 \subset \mathbb{R}^{N_2}$, $Z_2 \subset \partial G_2$. If the divergence theorem holds for G_1, $\partial G_1 \setminus Z_1$ and for G_2, $\partial G_2 \setminus Z_2$, then it holds for G, $\partial G \setminus Z$ where $G = G_1 \times G_2 \subset \mathbb{R}^{N_1+N_2}$ and $\partial G \setminus Z = \big((\partial G_1 \setminus Z_1) \times G_2 \big) \uplus \big(G_1 \times (\partial G_2 \setminus Z_2) \big)$.

Prove it.[2]

An N-box is the product of N intervals, of course. Also, a cylinder $\{(x, y, z) : x^2 + y^2 < r^2, 0 < z < a\}$ is the product of a disk and an interval.

4c Divergence of gradient: Laplacian

Some (but not all) vector fields are gradients of scalar fields.

4c1 Definition. (a) The *Laplacian* Δf of a function $f \in C^2(G)$ on an open set $G \subset \mathbb{R}^n$ is

$$\Delta f = \operatorname{div} \nabla f \,.$$

(b) f is *harmonic,* if $\Delta f = 0$.

We have $\nabla f = (D_1 f, \dots, D_n f)$, thus, $\operatorname{div} \nabla f = D_1(D_1 f) + \dots + D_n(D_n f)$; in this sense,

$$\Delta = D_1^2 + \dots + D_n^2 = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \,,$$

the so-called Laplace operator, or Laplacian.

Any n-dimensional Euclidean space may be used instead of $\mathbb{R}^n$. Indeed, the gradient is well-defined in such space, and the divergence is well-defined even without Euclidean metric.

The divergence theorem 4a3 gives, for a smooth G, the so-called *first Green formula*

$$\int_G \Delta f = \int_{\partial G} \langle \nabla f, \mathbf{n} \rangle = \int_{\partial G} D_{\mathbf{n}} f \,, \tag{4c2}$$

[1] Not a standard terminology.

[2] Hint: $\operatorname{div} F = (D_1 F_1 + \dots + D_{N_1} F_{N_1}) + (D_{N_1+1} F_{N_1+1} + \dots + D_{N_1+N_2} F_{N_1+N_2})$.

where $\big(D_{\mathbf{n}}f\big)(x) = \big(D_{\mathbf{n}_x}f\big)_x$ is the directional derivative of f at x in the normal direction $\mathbf{n}_x$. Here $f \in C^2(G)$, with bounded second derivatives.

Here is another instance of integration by parts. Let $u \in C^1(G)$, with bounded gradient, and $v \in C^2(G)$, with bounded second derivatives. Applying (4a5) to $f = u$ and $F = \nabla v$ we get $\int_G \langle \nabla u, \nabla v \rangle = \int_{\partial G} u \langle \nabla v, \mathbf{n} \rangle - \int_G u \Delta v$, that is,

(4c3) $$\int_G (u\Delta v + \langle \nabla u, \nabla v \rangle) = \int_{\partial G} \langle u \nabla v, \mathbf{n} \rangle = \int_{\partial G} u D_{\mathbf{n}} v \,,$$

the *second Green formula.* It follows that

(4c4) $$\int_G (u\Delta v - v\Delta u) = \int_{\partial G} (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u) \,,$$

the *third Green formula;* here $u, v \in C^2(G)$, with bounded second derivatives. In particular,

$$\int_{\partial G} u D_{\mathbf{n}} v = \int_{\partial G} v D_{\mathbf{n}} u \quad \text{for harmonic } u, v \,.$$

Rewriting (4c4) as

(4c5) $$\int_G u \Delta v = \int_G v \Delta u - \int_{\partial G} v D_{\mathbf{n}} u + \int_{\partial G} (D_{\mathbf{n}} v) u$$

we may say that really $\int (u 1\!\!1_G) \Delta v = \int v \Delta (u 1\!\!1_G)$ where $\Delta(u 1\!\!1_G)$ consists of the usual Laplacian $(\Delta u) 1\!\!1_G$ sitting on G and the singular Laplacian sitting on ∂G, of two terms, so-called single layer $(-D_{\mathbf{n}} u)$ and double layer $u D_{\mathbf{n}}$. Why two layers? Because the Laplacian (unlike gradient and divergence) involves second derivatives.

4c6 Exercise. Consider homogeneous polynomials on $\mathbb{R}^2$:

$$f(x, y) = \sum_{k=0}^m c_k x^k y^{m-k} \,.$$

For $m = 1, 2$ and 3 find all harmonic functions among these polynomials.[1]

4c7 Exercise. On $\mathbb{R}^2$,

(a) a function of the form

$$f(x, y) = \sum_{k=1}^m c_k \mathrm{e}^{a_k x + b_k y} \quad (a_k, b_k, c_k \in \mathbb{R})$$

[1] In fact, they are $\operatorname{Re}(x + \mathrm{i}y)^m$, $\operatorname{Im}(x + \mathrm{i}y)^m$ and their linear combinations.

is harmonic only if it is constant;

(b) a function of the form

$$f(x,y) = \mathrm{e}^{ax} \cos by$$

is harmonic if and only if $|a| = |b|$.[1]
Prove it.

Now, what about a radial harmonic function? We seek a radial f such that ∇f is of zero divergence, that is, $\nabla f(x) = \frac{\mathrm{const}}{|x|^N} x$ (recall (4a7)). By 4a6(a), $f(x) = g(|x|)$ where $\frac{g'(r)}{r} = \frac{\mathrm{const}}{r^N}$; thus, $g(r) = \frac{\mathrm{const}_1}{r^{N-2}} + \mathrm{const}_2$ for $N \ne 2$. We choose

$$f(x) = \frac{1}{|x|^{N-2}} \,; \quad \Delta f(x) = 0 \quad \text{for } x \ne 0 \,. \tag{4c8}$$

(This works also for $N = 1$: $f(x) = |x|$ is harmonic on $\mathbb{R} \setminus \{0\}$.) But for $N = 2$ we get $g'(r) = \frac{\mathrm{const}}{r}$; $g(r) = \mathrm{const}_1 \cdot \log r + \mathrm{const}_2$; we choose

$$f(x) = -\log|x| = \log\frac{1}{|x|} \,; \quad \Delta f(x) = 0 \quad \text{for } x \ne 0 \,. \tag{4c9}$$

The flux of ∇f through a sphere is[2]

$$\int_{|x|=r} D_{\mathbf{n}} f = \begin{cases} -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} & \text{for } N \ne 2, \\ -2\pi & \text{for } N = 2; \end{cases}$$

and, similarly to (4a9), the same holds for every smooth set $G \ni 0$.

4d Laplacian at a singular point

The function $g(x) = 1/|x|^{N-2}$ is harmonic on $\mathbb{R}^N \setminus \{0\}$, thus, for every $f \in C^2$ compactly supported within $\mathbb{R}^N \setminus \{0\}$,

$$\int g\Delta f = \int f \Delta g = 0 \,.$$

It appears that for $f \in C^2(\mathbb{R}^N)$ with a compact support,

$$\int g\Delta f = \mathrm{const} \cdot f(0) \,;$$

in this sense g has a kind of singular Laplacian at the origin.

[1] That is, $f(x,y) = \mathrm{Re}\left(\mathrm{e}^{x+\mathrm{i}y}\right)$.

[2] $\mathrm{const} = -(N-2)\mathrm{const}_1 = -(N-2)$ for $N \ne 2$, and $\mathrm{const} = \mathrm{const}_1 = -1$ for $N = 2$.

4d1 Lemma.

$$\int_{\mathbb{R}^N} \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0)$$

for every $N > 2$ and $f \in C^2(\mathbb{R}^N)$ with a compact support.

This improper integral converges, since $1/|x|^{N-2}$ is improperly integrable near 0. The coefficient $\frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the $(N-1)$-dimensional volume of the unit sphere (recall (3c9)).

Proof. For arbitrary $\varepsilon > 0$ we consider the function $g_\varepsilon(x) = 1/\big(\max(|x|, \varepsilon)\big)^{N-2}$, and $g(x) = 1/|x|^{N-2}$. Clearly, $\int |g_\varepsilon - g| \to 0$ (as $\varepsilon \to 0$), and $\int |g_\varepsilon - g||\Delta f| \to 0$, thus, $\int g_\varepsilon \Delta f \to \int g \Delta f$. We take $R \in (0, \infty)$ such that $f(x) = 0$ for $|x| \ge R$, introduce smooth sets $G_1 = \{x : |x| < \varepsilon\}$, $G_2 = \{x : \varepsilon < |x| < R\}$, and apply (4c4), taking into account that $\Delta g_\varepsilon = 0$ on G_1 and G_2:

$$\int g_\varepsilon \Delta f = \bigg(\int_{G_1} + \int_{G_2} \bigg) g_\varepsilon \Delta f = \bigg(\int_{\partial G_1} + \int_{\partial G_2} \bigg) \big(g_\varepsilon D_{\mathbf{n}} f - f D_{\mathbf{n}} g_\varepsilon \big) \, ;$$

however, these $D_{\mathbf{n}}$ must be interpreted differently under $\int_{\partial G_1}$ and $\int_{\partial G_2}$:

$$\int_{\partial G_1} g_\varepsilon D_{\mathbf{n}_1} f = \int_{|x|=\varepsilon} \frac{1}{\varepsilon^{N-2}} D_{\mathbf{n}} f \, ,$$
$$\int_{\partial G_2} g_\varepsilon D_{\mathbf{n}_2} f = \int_{|x|=\varepsilon} \frac{1}{\varepsilon^{N-2}} D_{-\mathbf{n}} f$$

where $\mathbf{n}$ is the outward normal of G_1 and inward normal of G_2; these two summands cancel each other. Further, $\int_{\partial G_1} f D_{\mathbf{n}_1} g_\varepsilon = \int_{|x|=\varepsilon} f \cdot 0 = 0$ since g_ε is constant on G_1; and

$$\int_{\partial G_2} f D_{\mathbf{n}_2} g_\varepsilon = \int_{|x|=\varepsilon} f \cdot \frac{N-2}{\varepsilon^{N-1}} \, ,$$

since $g_\varepsilon(x) = 1/|x|^{N-2}$ on G_2, and $f(x) = 0$ when $|x| = R$. Finally,

$$\int g_\varepsilon \Delta f = -(N-2) \frac{1}{\varepsilon^{N-1}} \int_{|x|=\varepsilon} f = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f_\varepsilon \, ,$$

where f_ε is the mean value of f on the ε-sphere. By continuity, $f_\varepsilon \to f(0)$ as $\varepsilon \to 0$; and, as we know, $\int g_\varepsilon \Delta f \to \int g \Delta f$. □

4d2 Remark. For $N = 2$ the situation is similar:

$$\int_{\mathbb{R}^2} \Delta f(x) \log \frac{1}{|x|} \, \mathrm{d}x = -2\pi f(0)$$

for every compactly supported $f \in C^2(\mathbb{R}^2)$.

When the boundary consists of a hypersurface and an isolated point, we get a combination of (4c5) and 4d1: a singular point and two layers.

4d3 Remark. Let $G \subset \mathbb{R}^N$ be a smooth set, $f \in C^2(G)$ with bounded second derivatives, and $0 \in G$. Then

$$\int_G \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) - \\ - \int_{\partial G} \Big(x \mapsto f(x) D_{\mathbf{n}} \frac{1}{|x|^{N-2}} \Big) + \int_{\partial G} \Big(x \mapsto (D_{\mathbf{n}} f(x)) \frac{1}{|x|^{N-2}} \Big) \, .$$

The proof is very close to that of 4d1. The case $N = 2$ is similar to 4d2, of course.

The case $G = \{x : |x| < R\}$ is especially interesting. Here $\partial G = \{x : |x| = R\}$; on ∂G,

$$\frac{1}{|x|^{N-2}} = \frac{1}{R^{N-2}} \quad \text{and} \quad D_{\mathbf{n}_x} \frac{1}{|x|^{N-2}} = -\frac{N-2}{R^{N-1}} \, ;$$

thus,

$$\int_{|x|<R} \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) + \frac{N-2}{R^{N-1}} \int_{|\cdot|=R} f + \frac{1}{R^{N-2}} \int_{|\cdot|=R} D_{\mathbf{n}} f \, .$$

Taking into account that $\int_{|\cdot|=R} D_{\mathbf{n}} f = \int_{|\cdot|<R} \Delta f$ by (4c2) we get

$$(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) = -\int_{|x|<R} \Big(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}} \Big) \Delta f(x) \, \mathrm{d}x + \frac{N-2}{R^{N-1}} \int_{|\cdot|=R} f$$

for $N > 2$; and similarly,

$$2\pi f(0) = -\int_{|x|<R} \big(\log R - \log |x| \big) \Delta f(x) \, \mathrm{d}x + \frac{1}{R} \int_{|\cdot|=R} f$$

for $N = 2$. In particular, for a harmonic f,

$$f(0) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{1}{R^{N-1}} \int_{|\cdot|=R} f = \frac{\int_{|\cdot|=R} f}{\int_{|\cdot|=R} 1}$$

for $N \ge 2$; the following result is thus proved (and holds also for $N = 1$, trivially).

4d4 Proposition (*Mean value property*)**.** For every harmonic function on a ball, with bounded second derivatives, its value at the center of the ball is equal to its mean value on the boundary of the ball.[1]

4d5 Remark. Now it is easy to understand why harmonic functions occur in physics ("the stationary heat equation"). Consider a homogeneous material solid body (in three dimensions). Fix the temperature on its boundary, and let the heat flow until a stationary state is reached. Then the temperature in the interior is a harmonic function (with the given boundary conditions).

4d6 Remark. Can the mean value property be generalized to a non-spherical boundary? We leave this question to more special courses (PDE, potential theory). But here is the idea. In 4d3 we may replace $\int_G \frac{\Delta f(x)}{|x|^{N-2}}\,\mathrm{d}x$ with $\int_G \Big(\frac{1}{|x|^{N-2}} + g(x) \Big) \Delta f(x)\,\mathrm{d}x$ where g is a harmonic function satisfying $\frac{1}{|x|^{N-2}} + g(x) = 0$ for all $x \in \partial G$ (if we are lucky to have such g). Then the double layer $\int_{\partial G} (D_{\mathbf{n}} v) u$ in (4c5), and the corresponding term in 4d3, disappears, and we get

$$(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) = \int_{\partial G} \bigg(x \mapsto f(x) D_{\mathbf{n}} \Big(\frac{1}{|x|^{N-2}} + g(x) \Big) \bigg) .$$

4d7 Exercise (*Maximum principle for harmonic functions*)**.**
Let u be a harmonic function on a connected open set $G \subset \mathbb{R}^N$. If $\sup_{x\in G} u(x) = u(x_0)$ for some $x_0 \in G$ then u is constant.

Prove it.[2]

It appears that

$$\Delta f(x) = 2N \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \Big(\big(\text{mean of } f \text{ on } \{ y : |y-x| = \varepsilon \} \big) - f(x) \Big) . \tag{4d8}$$

4d9 Exercise. (a) Prove that, for $N > 2$,

$$\frac{1}{R^2} \int_{|x|<R} \Big(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}} \Big) \,\mathrm{d}x \quad \text{does not depend on } R;$$

and for $N = 2$, $\frac{1}{R^2} \int_{|x|<R} \big(\log R - \log |x| \big) \,\mathrm{d}x$ does not depend on R. (No need to calculate these integrals.)[3]

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[1] In fact, the mean value property is also sufficient for harmonicity, even if differentiability is not assumed.

[2] Hint: the set $\{ x_0 : u(x_0) = \sup_{x\in G} u(x) \}$ is both open and closed in G.

[3] Hint: change of variable.

(b) For f of class C^2 near the origin, prove that the mean value of f on $\{x : |x| = \varepsilon\}$ is $f(0) + c_N\varepsilon^2\Delta f(0) + o(\varepsilon^2)$ as $\varepsilon \to 0$, for some $c_2, c_3, \dots \in \mathbb{R}$ (not dependent on f).

(c) Applying (b) to $f(x) = |x|^2$, find $c_2, c_3, \dots$ and prove (4d8).

4d10 Exercise. (a) For every f integrable (properly) on $\{x : |x| < R\}$,

$$\frac{\int_{|\cdot|<R} f}{\int_{|\cdot|<R} 1} = \int_0^R \frac{\int_{|\cdot|=r} f}{\int_{|\cdot|=r} 1} \frac{\mathrm{d}r^N}{R^N} \, .$$

(b) For every bounded harmonic function on a ball, its value at the center of the ball is equal to its mean value on the ball.

Prove it.[1]

4d11 Proposition. *(Liouville's theorem for harmonic functions)*

Every harmonic function $\mathbb{R}^N \to [0, \infty)$ is constant.

Proof. For arbitrary $x, y \in \mathbb{R}^N$ and $R > 0$ we have

$$f(x) = \frac{\int_{|z-x|<R} f(z)\,\mathrm{d}z}{\int_{|z-x|<R} \mathrm{d}z} \le \frac{\int_{|z-y|<R+|x-y|} f(z)\,\mathrm{d}z}{\int_{|z-x|<R} \mathrm{d}z} =$$
$$= \Big(\frac{R+|x-y|}{R}\Big)^N \frac{\int_{|z-y|<R+|x-y|} f(z)\,\mathrm{d}z}{\int_{|z-y|<R+|x-y|} \mathrm{d}z} = \Big(\frac{R+|x-y|}{R}\Big)^N f(y) \, ,$$

since the R-neighborhood of x is contained in the $(R+|x-y|)$-neighborhood of y. In the limit $R \to \infty$ we get $f(x) \le f(y)$; similarly, $f(y) \le f(x)$. □

4e Differential forms of order $N - 1$

It is easy to generalize the flux, defined by 4a1, as follow.

$n = N - 1$

4e1 Definition. Let $M \subset \mathbb{R}^N$ be an n-manifold,[2] $F : M \to \mathbb{R}^N$ a mapping continuous almost everywhere, and $\mathbf{n} : M \to \mathbb{R}^N$ a continuous mapping such that $\mathbf{n}_x$ is a unit normal vector to M at x, for each $x \in M$. The *flux* of (the vector field) F through (the hypersurface) M in the direction $\mathbf{n}$ is

$$\int_M \langle F, \mathbf{n} \rangle \, .$$

(The integral is treated as improper, and may converge or diverge.)

[1] Hint: (a) recall 13c8.

[2] Necessarily orientable; see 4e9.

It is not easy to calculate this integral, even if M is single-chart; the formula is complicated,

$$\int_M \langle F, \mathbf{n} \rangle = \int_G \langle F(\psi(u)), \mathbf{n}_{\psi(u)} \rangle \sqrt{\det(\langle (D_i\psi)_u, (D_j\psi)_u \rangle)_{i,j}} \, \mathrm{d}u \,,$$

and still, $\mathbf{n}_x$ should be calculated somehow. Fortunately, there is a better formula:[1] $n{=}N{-}1$

$$\text{(4e2)} \qquad \int_M \langle F, \mathbf{n} \rangle = \pm \int_G \det\bigl(F(\psi(u)), (D_1\psi)_u, \dots, (D_n\psi)_u \bigr) \, \mathrm{d}u$$

(and the sign $\pm$ will be clarified soon). That is, $\int_M \langle F, \mathbf{n} \rangle = \pm \int_M \omega$, where ω is the n-form defined by $\omega(x, h_1, \dots, h_n) = \det\bigl(F(x), h_1, \dots, h_n\bigr)$. We have to understand better this relation between vector fields and differential forms.

Recall two types of integral over an n-manifold:

* of an n-form ω, $\int_{(M,\mathcal{O})} \omega$, defined by (2c2) and (2d4);
* of a function f, $\int_M f$, defined by (2d8) and (2d9);

they are related by

$$\int_M f = \int_{(M,\mathcal{O})} f \mu_{(M,\mathcal{O})} \,,$$

where $\mu_{(M,\mathcal{O})}$ is the volume form; that is, $\int_M f = \int_{(M,\mathcal{O})} \omega$ where $\omega = f \mu_{(M,\mathcal{O})}$. Interestingly, every n-form ω on an orientable n-manifold $M \subset \mathbb{R}^N$ is $f \mu_{(M,\mathcal{O})}$ for some $f \in C(M)$. This is a consequence of the one-dimensionality[2] of the space of all antisymmetric multilinear n-forms on the tangent space T_xM. We have $f(x) = \omega(x, e_1, \dots, e_n)$ for some (therefore, every) orthonormal basis $(e_1, \dots, e_n)$ of T_xM that conforms to $\mathcal{O}_x$. But if ω is defined on the whole $\mathbb{R}^N$ (not just on M), it does not lead to a function f on the whole $\mathbb{R}^N$; indeed, in order to find $f(x)$ we need not just x but also T_xM (and its orientation).

The case $n = N$ is simple: every N-form ω on $\mathbb{R}^N$ (or on an open subset of $\mathbb{R}^N$) is $f \det$ (for some continuous f); here "det" denotes the volume form on $\mathbb{R}^N$; that is,

$$\text{(4e3)} \qquad \begin{aligned} \omega(x, h_1, \dots, h_N) &= f(x) \det(h_1, \dots, h_N) \,; \\ f(x) &= \omega(x, e_1, \dots, e_N) \,. \end{aligned}$$

[1] A wonder: the volume form of M is not needed; the volume form of $\mathbb{R}^N$ (the determinant) is used instead. Why so? Since the flux is the *volume* of fluid flowing through the surface (per unit time), as was noted in 4a.

[2] Recall Sect. 1e and 2c.

Note that for every open $U \subset \mathbb{R}^N$,

$$\int_U f \det = \int_U f(x)\,\mathrm{d}x\,; \qquad \int_U \det = v(U)\,. \tag{4e4}$$

We turn to the case $n = N - 1$.

The space of all antisymmetric multilinear n-forms L on $\mathbb{R}^N$ is of dimension $\binom{N}{n} = N$. Here is a useful linear one-to-one correspondence between such L and vectors $h \in \mathbb{R}^N$: $n\!=\!N\!-\!1$

$$\forall h_1, \dots, h_n \;\; L(h_1, \dots, h_n) = \det(h, h_1, \dots, h_n)\,.$$

Introducing the cross-product $h_1 \times \dots \times h_n$ by[1]

$$\forall h \;\; \langle h, h_1 \times \dots \times h_n \rangle = \det(h, h_1, \dots, h_n) \tag{4e5}$$

(it is a vector orthogonal to $h_1, \dots, h_n$), we get

$$L(h_1, \dots, h_n) = \langle h, h_1 \times \dots \times h_n \rangle\,.$$

Doing so at every point, we get a linear one-to-one correspondence between n-forms ω on $\mathbb{R}^N$ and (continuous) vector fields F on $\mathbb{R}^N$:

$$\omega(x, h_1, \dots, h_n) = \langle F(x), h_1 \times \dots \times h_n \rangle = \det(F(x), h_1, \dots, h_n)\,. \tag{4e6}$$

Similarly, $(n-1)$-forms ω on an oriented n-dimensional manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ (not just $N - n = 1$) are in a linear one-to-one correspondence with *tangent* vector fields F on M, that is, $F \in C(M \to \mathbb{R}^N)$ such that $\forall x \in M \;\; F(x) \in T_x M$. $n\!=\!N\!-\!1$

Let $M \subset \mathbb{R}^N$ be an orientable n-manifold, ω and F as in (4e6). We know that $\omega|_M = f \mu_{(M,\mathcal{O})}$ for some f. How is f related to F? Given $x \in M$, we take an orthonormal basis $(e_1, \dots, e_n)$ of $T_x M$, note that $e_1 \times \dots \times e_n = \mathbf{n}_x$ is a unit normal vector to M at x, and

$$\begin{aligned}\langle F(x), \mathbf{n}_x \rangle = \langle F(x), e_1 \times \dots \times e_n \rangle = \omega(x, e_1, \dots, e_n) = \\ = f(x) \mu_{(M,\mathcal{O})}(x, e_1, \dots, e_n) = \pm f(x)\,.\end{aligned}$$

In order to get "+" rather than "±" we need a coordination between the orientation $\mathcal{O}$ and the normal vector $\mathbf{n}_x$. Let the basis $(e_1, \dots, e_n)$ of $T_x M$

[1]For $N = 3$ the cross-product is a binary operation, but for $N > 3$ it is not. In fact, it is possible to define the corresponding associative binary operation (the so-called exterior product, or wedge product), not on vectors but on the so-called multivectors, see "Multivector" and "Exterior algebra" in Wikipedia.

conform to the orientation $\mathcal{O}_x$ (of M at x, or equivalently, of T_xM, recall Sect. 2b), then $\mu_{(M,\mathcal{O})}(x, e_1, \dots, e_n) = +1$. The two unit normal vectors being $\pm e_1 \times \dots \times e_n$, we say that $\mathbf{n}_x = e_1 \times \dots \times e_n$ conforms to the given orientation, and get[1]

$$\langle F(x), \mathbf{n}_x \rangle = f(x) ; \quad \omega|_M = \langle F, \mathbf{n} \rangle \mu_{(M,\mathcal{O})} .$$

Integrating this over M, we get nothing but the flux! Recall 4e1: the flux of F through M is $\int_M \langle F, \mathbf{n} \rangle$, that is, $\int_{(M,\mathcal{O})} \langle F, \mathbf{n} \rangle \mu_{(M,\mathcal{O})} = \int_{(M,\mathcal{O})} \omega|_M = \int_{(M,\mathcal{O})} \omega$. We get (4e2), and moreover,

$$\int_M \langle F, \mathbf{n} \rangle = \int_{(M,\mathcal{O})} \omega \tag{4e7}$$

for ω of (4e6) and $\mathcal{O}$ conforming to $\mathbf{n}$. In particular, when M is single-chart, we have

$$\int_M \langle F, \mathbf{n} \rangle = \int_G \det\bigl(F(\psi(u)), (D_1\psi)_u, \dots, (D_n\psi)_u \bigr) \, \mathrm{d}u \tag{4e8}$$

provided that $\det(\mathbf{n}, D_1\psi, \dots, D_n\psi) > 0$. Necessarily, $D_1\psi \times \dots \times D_n\psi = c\mathbf{n}$ for some $c \ne 0$ (since both vectors are orthogonal to the tangent space); the sign of c is the sign in (4e2).

We summarize the situation with the sign.

$n = N-1$

4e9 Remark. For an n-dimensional manifold $M \subset \mathbb{R}^N$, the two orientations $\mathcal{O}_x$ at a given point $x \in M$ correspond naturally[2] to the two unit normal vectors $\mathbf{n}_x$ to M at x. Namely, for some (therefore, every) orthonormal basis $e_1, \dots, e_n$ of T_xM that conforms to $\mathcal{O}_x$,

(a) $\det(\mathbf{n}_x, e_1, \dots, e_n) = +1$;

or, equivalently,

(b) $e_1 \times \dots \times e_n = \mathbf{n}_x$.

Alternatively (and equivalently), for arbitrary (not just orthonormal) basis,

(a′) $\det(\mathbf{n}_x, e_1, \dots, e_n) > 0$;

(b′) $e_1 \times \dots \times e_n = c\mathbf{n}_x$ for some $c > 0$.

Given a chart (G, ψ) of M around x that conforms to $\mathcal{O}_x$, we may take $e_i = (D_i\psi)_{\psi^{-1}(x)}$.

Orientations $(\mathcal{O}_x)_{x \in M}$ of M correspond naturally to continuous mappings $M \ni x \mapsto \mathbf{n}_x \in \mathbb{R}^N$ such that for every $x \in M$, $\mathbf{n}_x$ is a unit normal vector to M at x. Thus, such mappings exist if and only if M is orientable (and in this case, there are exactly two of them, provided that M is connected).

[1] Not unexpectedly, in order to find $f(x)$ we need not just x but also $\mathbf{n}_x$.

[2] Using the orientation of $\mathbb{R}^N$ given by the determinant; the other orientation of $\mathbb{R}^N$ leads to the other correspondence.

We turn to a smooth set $U \subset \mathbb{R}^N$. Its boundary ∂U is a hypersurface; the outward normal vector leads, according to 4e9, to an orientation of ∂U. *In such cases we always use this orientation.* Given $F \in C^1(U \to \mathbb{R}^N)$ with DF bounded, we may rewrite the divergence theorem 4a3, $\int_U \operatorname{div} F = \int_{\partial U} \langle F, \mathbf{n} \rangle$, as

$$\int_U (\operatorname{div} F) \det = \int_{\partial U} \omega$$

where ω corresponds to F according to (4e6). Taking into account that every n-form of class C^1 corresponds to some vector field, we conclude.

$n = N-1$

4e10 Proposition. For every n-form ω of class C^1 on $\mathbb{R}^N$ there exists an N-form ω' on $\mathbb{R}^N$ such that for every smooth set $U \subset \mathbb{R}^N$,

$$\int_{\partial U} \omega = \int_U \omega' \,.$$

4e11 Remark. The same holds in the piecewise smooth case: $\int_{\partial U \setminus Z} \omega = \int_U \omega'$ provided that the divergence theorem holds for U and $\partial U \setminus Z$.

4e12 Example. On $\mathbb{R}^2$ consider a vector field $F : \binom{x}{y} \mapsto \binom{F_1(x,y)}{F_2(x,y)}$ and a curve (1-manifold) covered by a single chart $\psi : (a,b) \to \mathbb{R}^2$, $\psi(t) = \binom{\psi_1(t)}{\psi_2(t)}$. Using the complicated formula,

$$\mathbf{n}_{\psi(t)} = \frac{1}{\sqrt{\psi_1'^2(t) + \psi_2'^2(t)}} \begin{pmatrix} \psi_2'(t) \\ -\psi_1'(t) \end{pmatrix} ; \quad J_\psi(t) = \sqrt{\psi_1'^2(t) + \psi_2'^2(t)} \,;$$

$$\langle F(\psi(t)), \mathbf{n}_{\psi(t)} \rangle = \frac{1}{\sqrt{\dots}} \big(F_1 \psi_2' - F_2 \psi_1' \big) \,;$$

$$\text{flux} = \int_a^b \langle F(\psi(t)), \mathbf{n}_{\psi(t)} \rangle J_\psi(t) \, \mathrm{d}t = \int_a^b \big(F_1 \psi_2' - F_2 \psi_1' \big) \, \mathrm{d}t \,.$$

Alternatively, using (4e8),

$$\det\big(F(\psi(t)), \psi'(t)\big) = \begin{vmatrix} F_1 & \psi_1' \\ F_2 & \psi_2' \end{vmatrix} = F_1 \psi_2' - F_2 \psi_1' \,; \quad \text{flux} = \int_a^b \big(F_1 \psi_2' - F_2 \psi_1' \big) \, \mathrm{d}t \,.$$

4e13 Exercise. Fill in the details in 4e12.

4e14 Example. Continuing 4e12, consider the 1-form ω, $\omega\big(\binom{x}{y}, \binom{dx}{dy}\big) = f_1(x,y)\,dx + f_2(x,y)\,dy$; it corresponds to F according to (4e6) when

$$f_1(x,y)\,dx + f_2(x,y)\,dy = \begin{vmatrix} F_1(x,y) & dx \\ F_2(x,y) & dy \end{vmatrix} \,, \quad \text{that is,} \quad \begin{aligned} f_1 &= -F_2 \,, \\ f_2 &= F_1 \,. \end{aligned}$$

In this case,

$$\int_M \omega = \int_a^b \omega\big(\psi(t), \psi'(t)\big)\,\mathrm{d}t = \int_a^b \big(f_1(\psi(t))\psi_1'(t) + f_2(\psi(t))\psi_2'(t)\big)\,\mathrm{d}t = \\ = \int_a^b \big(-F_2\psi_1' + F_1\psi_2'\big)\,\mathrm{d}t = \text{flux}\,.$$

4e15 Exercise. Fill in the details in 4e14.

4e16 Remark. Less formally, denoting $\psi_1(t)$ and $\psi_2(t)$ by just $x(t)$ and $y(t)$ we have

$$\int_M \omega = \int_a^b \big(f_1(x(t), y(t))x'(t) + f_2(x(t), y(t))y'(t)\big)\,\mathrm{d}t\,;$$

naturally, this is called $\int_M (f_1\,dx + f_2\,dy)$.

4e17 Example. Continuing 4e12 and 4e14, we calculate the divergence:

$$\operatorname{div} F = D_1F_1 + D_2F_2 = D_1f_2 - D_2f_1\,;$$

thus,

$$\omega' = (\operatorname{div} F)\det = (D_1f_2 - D_2f_1)\det\,;$$
$$\int_{\partial U} \omega = \int_U (D_1f_2 - D_2f_1)$$

for a smooth $U \subset \mathbb{R}^2$. If ∂U is covered (except for a single point) with a chart $\psi : (a, b) \to \mathbb{R}^2$, $\psi(a+) = \psi(b-)$, then 4e10 gives

$$\int_{\partial U} (f_1\,dx + f_2\,dy) = \int_U (D_1f_2 - D_2f_1)\,.$$

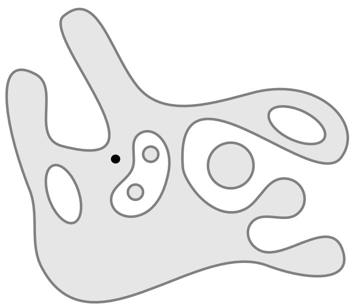
This is the well-known Green's theorem; in traditional notation,

$$\oint_{\partial U} (L\,dx + M\,dy) = \iint_U \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathrm{d}x\mathrm{d}y\,.$$

4e18 Example. The 1-form $\omega = \frac{-y\,dx + x\,dy}{2}$ on $\mathbb{R}^2$ (mentioned in Sect. 1d) corresponds to the vector field $F\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big) = \frac12\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big)$, that is, $F(x) = \frac12 x$ for $x \in \mathbb{R}^2$. Clearly, $\operatorname{div} F = 1$, thus, $\omega' = \det$; by 4e10,

$$\int_{\partial U} \omega = v(U) \quad \text{for every smooth } U \subset \mathbb{R}^2\,.$$

4e19 Example.



The 1-form $\omega = \frac{-y\,dx+x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ (treated in Sect. 1d) corresponds to the vector field $F\binom{x}{y} = \frac{1}{x^2+y^2}\binom{x}{y}$, that is, $F(x) = \frac{x}{|x|^2}$ for $x \in \mathbb{R}^2 \setminus \{0\}$. By (4a7), $\operatorname{div} F = 0$ on $\mathbb{R}^2 \setminus \{0\}$, thus $\omega' = 0$ on $\mathbb{R}^2 \setminus \{0\}$; by 4e10, $\int_{\partial U} \omega = 0$ for every smooth U such that $\overline{U} \not\ni 0$. On the other hand, for every smooth $U \ni 0$ we have $\int_{\partial U} \omega = 2\pi$ by (4a9); compare this fact with Sect. 1d.

Similarly, in $\mathbb{R}^3$ the 2-form ω that corresponds to the vector field $F(x) = \frac{x}{|x|^3}$ satisfies $\int_{\partial U} \omega = 0$ whenever $\overline{U} \not\ni 0$, and $\int_{\partial U} \omega = 4\pi$ whenever $U \ni 0$.

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