Appendix to Sect. 6d: Better proof of “no isomorphism”.[1]

The complete partially ordered set (in fact, Boolean algebra) $\mathrm{BP}/\!\sim$ satisfies the following condition:

> There exists a countable set $S \subset \mathrm{BP}/\!\sim$ such that every element of $\mathrm{BP}/\!\sim$ is the supremum of some subset of S.

Equivalently:

> For every non-zero element of $\mathrm{BP}/\!\sim$ there exists a smaller non-zero element of S.

Proof. For every non-zero $[A] \in \mathrm{BP}/\!\sim$ the open set $U(A)$ contains some rational interval. □

In contrast, the set $\mathcal{A}_m/\overset{m}{\sim}$ violates this condition, which follows easily from the following lemma (and the Baire category theorem).

Lemma. For every non-zero $[B] \in \mathcal{A}_m/\overset{m}{\sim}$ the set of $[A] \in \mathcal{A}_m/\overset{m}{\sim}$ such that $[A] \ge [B]$ is nowhere dense.

Proof. The function $[A] \mapsto m(B \setminus A)$ is continuous (in fact, Lipschitz) and vanishes whenever $[A] \ge [B]$. It is > 0 on a dense (open) set, since B contains measurable subsets of arbitrarily small measure. □

[1] Proposed by Alon Titelman.