4 Choice axioms and Baire category theorem

4a Vitali set

The choice axiom implies existence of sets that are not Lebesgue measurable, as well as sets such that the corresponding Banach-Mazur game is not *determined,* that is, has no winning strategy, neither for Alice nor Bob. In fact, the same set can serve both cases.

Recall that a *Vitali set* is a set $V \subset [0,1)$ such that for every $x \in \mathbb{R}$ there exists one and only one $v \in V$ such that $x - v$ is rational. That is, V chooses one element in each equivalence class $\mathbb{Q} + x$ of the equivalence relation $x \sim y \iff x - y \in \mathbb{Q}$; here $\mathbb{Q}$ is the set of all rational numbers. Existence of such V follows immediately from the choice axiom.

Clearly, the (countably many of uncountable) sets $V + q$ for $q \in \mathbb{Q}$ are a partition of $\mathbb{R}$. In addition,

$$[0,1) \subset \bigcup_{q \in \mathbb{Q} \cap (-1,1)} V + q \subset (-1,2) \,.$$

It follows easily that V cannot be Lebesgue measurable. First, it cannot be a null set, since $\cup_{q \in \mathbb{Q} \cap (-1,1)} V + q$ is not a null set. Second, it cannot be a (measurable) set of positive measure, since $\cup_{q \in \mathbb{Q} \cap (-1,1)} V + q$ is not a set of infinite measure.

Similarly, V cannot be meager, since $\cup_{q \in \mathbb{Q} \cap (-1,1)} V + q$ is not meager.

4a1 Lemma. A Vitali set cannot be comeager in a nonempty open set.

Recall the relation $U \Vdash A$; note its evident properties: monotonicity,

$$\begin{aligned} U_1 \subset U_2 \,,\ \ U_2 \Vdash A \quad &\Longrightarrow \quad U_1 \Vdash A \,; \\ A_1 \subset A_2 \,,\ \ U \Vdash A_1 \quad &\Longrightarrow \quad U \Vdash A_2 \,; \end{aligned}$$

intersection,

$$U \Vdash A_1,\, U \Vdash A_2,\, \dots \quad \Longrightarrow \quad U \Vdash A_1 \cap A_2 \cap \dots\,,$$

invariance under homeomorphisms (in particular, shifts of $\mathbb{R}$), and a nonevident property: in a completely metrizable space

$$U \Vdash A \quad \Longrightarrow \quad A \ne \emptyset\,,$$

just another formulation of the Baire category theorem.

Proof of 4a1. Assume that $(a, a+2\varepsilon) \Vdash V$. Take $q \in \mathbb{Q} \cap (0, \varepsilon)$. On one hand, $(a+\varepsilon, a+2\varepsilon) \subset (a+q, a+2\varepsilon+q) \Vdash V+q$. On the other hand, $(a+\varepsilon, a+2\varepsilon) \subset (a, a+2\varepsilon) \Vdash V$. Thus, $(a+\varepsilon, a+2\varepsilon) \Vdash V \cap (V+q) = \emptyset$; a contradiction. □

By the way, a similar argument may be used when proving that V cannot be of positive measure. Indeed, for every set V of positive measure there exists an interval $(a, a+\varepsilon)$ such that $m\big(V \cap (a, a+\varepsilon)\big) \ge 0.9\varepsilon\dots$

We summarize.

4a2 Proposition. If V is a Vitali set then the corresponding Banach-Mazur game is not determined.

> The possibility of indeterminateness makes the Banach-Mazur game particularly interesting for the general theory of games. It also raises some interesting questions. If a game is determined in favor of one of the players, should it be called a game of "skill"? If neither player can control the outcome, is the outcome a matter of "chance"? What does "chance" mean in this connection?
>
> Oxtoby[1]

4b No choice

The Baire category theorem in a Polish[2] space can be proved without any choice axiom,[3] in ZF (the Zermelo-Fraenkel set theory).

First of all, a clarification (for non-experts in logic). Saying "this set is nonempty; we *choose* one of its elements and denote it by x" we do not use

[1] "Measure and category", near the end of Sect. 6.
[2] Not just completely metrizable!
[3] Why "any choice axiom" rather than "the choice axiom"? Wait for Sect. 4c...

any choice axiom. Rather, we use the first order logic (predicate calculus), namely, existential elimination (=instantiation):

$$\forall x \,\bigl(P(x) \implies Q\bigr) \;\&\; \exists x \, P(x) \quad \implies \quad Q\,.$$

Here is an example.

Claim: let X be a countable set; then there exists a function $f : 2^X \to X$ such that $f(A) \in A$ for all nonempty $A \subset X$. *Proof:* we *choose* a numbering of X, that is, a bijection $\varphi : X \to \{1, 2, \dots\}$ and define f by $f(A) = \varphi^{-1}\bigl(\min \varphi(A)\bigr)$. (No choice axiom needed.)

On the other hand, saying "each A_i is nonempty; we *choose* an element in each A_i and denote it by x_i" we need a choice axiom (unless i runs over a finite set). Here is an example.

Claim: the countable union of countable sets is a countable set. *Proof (sketch):* given countable X_n for $n = 1, 2, \dots$ we *choose* bijections $\varphi_n : X_n \to \{1, 2, \dots\}$ and note that the set $\{(n, x) : n = 1, 2, \dots; x \in X_n; n + \varphi_n(x) = M\}$ is finite for every $M = 2, 3, \dots$ (A choice axiom needed.)

Nevertheless the set of rational numbers is provably countable, still (with no choice); think, why.

So, a Polish space X is given, nowhere dense sets $A_1, A_2, \dots \subset X$ and a nonempty open set $U_0 \subset X$. We have to prove that $U_0 \setminus \cup_n A_n \ne \emptyset$ (with no choice).

Proof. We take a countable base $(U_n)_n$ of X (its existence, proved in 3a6, does not need a choice). We consider[1] the least n_1 such that $\operatorname{diam} U_{n_1} \le 2^{-1}$ and $\operatorname{Cl}(U_{n_1}) \subset U_0 \setminus \operatorname{Cl}(A_1)$. Then we consider the least n_2 such that $\operatorname{diam} U_{n_2} \le 2^{-2}$ and $\operatorname{Cl}(U_{n_2}) \subset U_{n_1} \setminus \operatorname{Cl}(A_2)$. And so on.[2] We get $(n_k)_k$, and $\cap_k U_{n_k} = \{x\}$, $x \in U_0 \setminus \cup_n A_n$. □

4c Dependent choice

The Baire category theorem in a completely metrizable[3] space can be proved with an axiom weaker than the choice axiom, the so-called dependent choice axiom.

Axiom of dependent choice (DC). Let A be a nonempty set and $R \subset A \times A$ satisfy $\forall a \in A \; \exists b \in A \;\; (a, b) \in R$. Then there exists an infinite sequence $(a_n)_n$ such that $(a_n, a_{n+1}) \in R$ for $n = 1, 2, \dots$.

[1] Not "choose"!

[2] By induction. It is crucial that no arbitrary choice is involved on n-th step.

[3] Generally, nonseparable.

Existence of a finite sequence of this kind is easy to prove by induction. Also, existence of an infinite sequence is evident if A is countable. However, DC is not provable in ZF.[1]

The choice axiom (AC) implies DC as follows: it gives a function $f : A \to A$ such that $\big(a, f(a)\big) \in R$ for all $a \in A$ (a "choice function"); we choose $a_1 \in A$ and define $(a_n)_n$ recursively: $a_{n+1} = f(a_n)$.

On the other hand, DC does not imply AC (that is, AC is not provable in ZF+DC unless the latter is inconsistent).

You may ask: why bother? The choice axiom is widely accepted in mathematics.

Here is my attitude. I do not doubt that AC is true, and still, I discriminate mathematical objects obtainable in ZF+DC (call them "tame") from these obtainable in ZFC (=ZF+AC) but not ZF+DC (call them "wild"). Note that I discriminate objects, not theorems. If a property of tame objects is proved using wild objects, it does not bother me. I only want to mark the wild objects by a warning "for internal use only".[2]

Mathematics provides models for other sciences. Normally, only tame objects are used in these models.[3] A Vitali set is wild. Recall also the Banach-Tarski paradox.

4c1 Lemma. (ZF+DC)[4] Let $A_1, A_2, \dots$ be nonempty sets and $R_n \subset A_n \times A_{n+1}$ satisfy $\forall a \in A_n \; \exists b \in A_{n+1} \;\; (a, b) \in R_n$ for $n = 1, 2, \dots$ Then there exists $(a_n)_n$ such that $(a_n, a_{n+1}) \in R_n$ for $n = 1, 2, \dots$

Proof. We consider the set A of all finite sequences $(a_1, \dots, a_n) \in A_1 \times \dots \times A_n$ (for all $n = 1, 2, \dots$) satisfying $(a_1, a_2) \in R_1, \dots, (a_{n-1}, a_n) \in R_{n-1}$. (For $n = 1$ these are just (a_1) for $a_1 \in A_1$.) We note that $(a_1, \dots, a_{n+1}) \in A$ implies $(a_1, \dots, a_n) \in A$, and consider the set $R \subset A \times A$ of all pairs of the form $\big((a_1, \dots, a_n), (a_1, \dots, a_{n+1})\big)$ in $A \times A$. DC gives (for *some* n) an infinite

[1] Unless ZF is inconsistent, of course. Proofs of such negative metamathematical claims are far beyond our course.

[2] Image from Scout's Honor co.

[3] For an attempt to use a "wild" object outside mathematics, see: I. Pitowsky (1982) "Resolution of the Einstein-Podolsky-Rosen and Bell paradoxes." Phys. Rev. Lett. **48**:19, 1299-1302. Not unexpectedly, the attempt failed; see: N.D. Mermin (1982) "Comment on 'Resolution of the Einstein-Podolsky-Rosen and Bell paradoxes'." Phys. Rev. Lett. **49**:16, 1214.

[4] It means, the lemma is proved in the theory ZF+DC.

sequence of finite sequences

$$\begin{aligned}&(a_1, \dots, a_n)\,, \\ &(a_1, \dots, a_n, a_{n+1})\,, \\ &(a_1, \dots, a_n, a_{n+1}, a_{n+2})\,, \\ &\dots\end{aligned}$$

satisfying $(a_k, a_{k+1}) \in R_k$ for all $k = 1, 2, \dots$ It remains to take $(a_k)_k$. $\Box$

4c2 Exercise. Prove in ZF+DC that

(a) if sets $A_1, A_2, \dots$ are nonempty then their product $A_1 \times A_2 \times \dots$ is nonempty;[1]

(b) if pairwise disjoint sets $A_1, A_2, \dots$ are countable then their union $A_1 \cup A_2 \cup \dots$ is countable.

4c3 Exercise. Prove in ZF+DC that

(a) if A_n, R_n are as in 4c1 and $a_1 \in A_1$ is given then there exist $a_2, a_3, \dots$ such that $(a_n, a_{n+1}) \in R_n$ for $n = 1, 2, \dots$;

(b) if A and R are as in Axiom of dependent choice, and $a \in A$ is given, then there exists $(a_n)_n$ such that $a_1 = a$ and $(a_n, a_{n+1}) \in R_n$ for $n = 1, 2, \dots$

4c4 Exercise. Prove in ZF+DC the Baire category theorem for a completely metrizable space.

4d The converse holds, but is not simple

Can the Baire category theorem for completely metrizable spaces be proved without DC (in ZF, or ZF plus some choice axiom weaker than DC)? No, it cannot, since the Baire theorem implies DC, as we'll see.

To this end we need to familiarize ourselves with (co)meager sets in non-separable spaces. (Till now we treated only separable spaces.) We really need only a straightforward generalization of the space $\{0,1\}^\infty$, the space S^∞ where S is an arbitrary set (maybe uncountable). We endow S with the discrete metric,

$$d(x, y) = \begin{cases} 1 & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

On the set S^∞ of all infinite sequences we want to introduce a metrizable topology such that, similarly to 1d2(b), the convergence is pointwise:

$$x_n \xrightarrow[n\to\infty]{} x \quad \Longleftrightarrow \quad \forall k \,\Big(x_n(k) \xrightarrow[n\to\infty]{} x(k) \Big)$$

[1] This is the so-called countable choice axiom, strictly weaker than DC but strictly stronger than nothing.

for $x, x_1, x_2, \dots \in S^\infty$; here "$x_n(k) \to x(k)$" means $d\big(x_n(k), x(k)\big) \to 0$, which means just $x_n(k) = x(k)$ for all n large enough (that is, $n \ge N_k$).

4d1 Exercise. Each of the following two metrics on S^∞ corresponds to the pointwise convergence:

(a) $\rho(x,y) = \sum_{k:x(k)\ne y(k)} 2^{-k}$;

(b) $\rho(x,y) = 1/\inf\{k : x(k) \ne y(k)\} = \sup_{k:x(k)\ne y(k)} \frac{1}{k} = \sup_k \frac{1}{k} d\big(x(k), y(k)\big)$.

Prove it.

Still, a neighborhood of x may be taken as $\{y : y(1) = x(1), \dots, y(n) = x(n)\}$; and 1d5 still applies.

4d2 Exercise. The space S^∞ is separable if and only if S is (at most) countable.

Usually $[0,1]^\infty$ denotes a separable (moreover, compact) metrizable space, since by default $[0,1]$ is endowed with its usual metric $|x-y|$, and accordingly, one of compatible metrics on $[0,1]^\infty$ is $\rho(x,y) = \sup_k \frac{1}{k}|x(k) - y(k)|$. An additional precaution is needed for $\mathbb{R}^\infty$, since $\mathbb{R}$ is not bounded; in this case $\rho(x,y) = \sup_k \frac{1}{k} \min\big(1, |x(k) - y(k)|\big)$. In order to avoid confusion, the corresponding nonseparable spaces will be denoted by $\big([0,1], d\big)^\infty$, $\big(\mathbb{R}, d\big)^\infty$ etc.

Such sets as $\{x : x(10) = 0\}$, $\{x : x(10) = x(13)\}$, $\{x : x(10) = \sin x(13)\}$ etc. are nowhere dense in $[0,1]^\infty$ or $\mathbb{R}^\infty$; however, in $\big([0,1], d\big)^\infty$ or $\big(\mathbb{R}, d\big)^\infty$ these sets are clopen (that is, both closed and open), not nowhere dense. By the way, continuity of sin is not relevant, since every map $(\mathbb{R}, d) \to (\mathbb{R}, d)$ is continuous.

Thus, a generic element of $\mathbb{R}^\infty$ is a sequence of pairwise distinct, non-zero numbers. The situation in $\big(\mathbb{R}, d\big)^\infty$ is strikingly different. First of all, the set $\{x : \forall k \;\; x(k) \ne 0\}$ is closed and nowhere dense! Just because it is the product of infinitely many clopen sets $\mathbb{R} \setminus \{0\} \subset (\mathbb{R}, d)$. (Recall the end of Sect. 2a.) Thus, a generic sequence of $(\mathbb{R}, d)^\infty$ contains 0, as well as 1, and π, etc. But this is only the tip of the iceberg.

4d3 Exercise. Formulate and prove for S^∞ a counterpart of 2a2. Do the same for 2a7.

4d4 Exercise. For every $s \in S$ the set $\{x \in S^\infty : \exists n \;\; x(n) = s\}$ is comeager. Prove it.

Imagine that we have a "nonseparable topological random number generator", a device able to produce a generic element of $(\mathbb{R}, d)^\infty$. Then we can solve the equation $f(x) = 0$ for an arbitrary $f : \mathbb{R} \to \mathbb{R}$ (not just continuous, not even measurable) provided that we are able to check the equality $f(x) = 0$ for any given x. Here is the know-how. We exercise the "nonseparable topological random number generator", getting $(x_n)_n$, $x_n \in \mathbb{R}$, and check the equalities $f(x_1) = 0$, $f(x_2) = 0$, ... until a solution is found. If the equation has a solution (at least one) then it should occur in the sequence!

But this is incredible. Unbelievable. Outrageous. The device does not know our question and still produces a countable subset containing the (possibly unique) answer!

Nothing like that can happen in probability theory. It may happen that for every $r \in \mathbb{R}$ a random set contains r almost surely. Then one applies Fubini's theorem and concludes that almost surely the random set is of full Lebesgue measure. That is, we have a measurable set $A \subset \mathbb{R} \times \Omega$ (Ω being a probability space) such that every section $A_r = \{\omega : (r, \omega) \in A\} \subset \Omega$ is of probability 1, and we conclude that almost every section $A^\omega = \{r : (r, \omega) \in A\} \subset \mathbb{R}$ is a set of full Lebesgue measure.

In contrast, consider the set $A \subset S \times S^\infty$ of all points (s, x) such that $\exists n \;\; x(n) = s$. This is a dense open set, and every section $A_s = \{x : (s, x) \in A\} \subset S^\infty$ is a dense open set (therefore, comeager). Nevertheless, every section $A^x = \{s : (s, x) \in A\} \subset S$ is ridiculously small: it is (at most) countable, while S is uncountable. Topologically, A^x is a clopen set, neither meager nor comeager.

On the other hand, it is not fair to compare the nonseparable topological theory with the probability theory. A sequence of independent random variables should be compared with $\mathbb{R}^\infty$, not $(\mathbb{R}, d)^\infty$. The discrete space $(\mathbb{R}, d)$ should be compared with the non-σ-finite measure space $(\mathbb{R}, \nu)$ where ν is the counting measure ($\nu(\{x\}) = 1$ for all x). But $(\mathbb{R}, \nu)$ is not at all a probability space. Thus, the nonseparable S^∞ has no probabilistic counterpart at all. Bearing this fact in mind we continue our excursion to this exotic space.

4d5 Exercise. Prove the following for a generic $(x_n)_n \in S^\infty$:
for every $s \in S$ the set $\{n : x(n) = s\}$ is either empty or infinite.[1]

It is convenient to use *category quantifiers:*

$\forall^* x$	for comeager many x for all generic x
$\exists^* x$	for non-meager many x for some generic x

[1] Rather strange for independent x_n... A physicist could say: like bosons?

That is,

$$\begin{aligned} \forall^* x \;\; x \in A \quad &\Longleftrightarrow \quad A \text{ is comeager}\,; \\ \exists^* x \;\; x \in A \quad &\Longleftrightarrow \quad A \text{ is not meager}\,. \end{aligned}$$

Also, for a nonempty open U,

$$\begin{aligned} \forall^* x \in U \;\; x \in A \quad &\Longleftrightarrow \quad U \Vdash A\,; \\ \exists^* x \in U \;\; x \in A \quad &\Longleftrightarrow \quad \neg \;\; U \Vdash (X \setminus A)\,. \end{aligned}$$

(Here X is the given space, and $\neg$ is negation.) Note that

$$\begin{aligned} \forall^* x \; (\dots) \quad &\Longleftrightarrow \quad \forall^* x \in X \; (\dots)\,; \\ \exists^* x \; (\dots) \quad &\Longleftrightarrow \quad \exists^* x \in X \; (\dots)\,; \\ \neg \;\; \exists^* x \in U \; (\dots) \quad &\Longleftrightarrow \quad \forall^* x \in U \; \neg \, (\dots)\,. \end{aligned}$$

Now 4d4 and 4d5 become

$$\begin{aligned} &\forall s \in S \; \forall^* x \in S^\infty \; \exists n \;\; x(n) = s\,; \\ &\forall^* x \in S^\infty \; \forall s \in S \;\; |\{n : x(n) = s\}| \in \{0, \infty\}\,. \end{aligned}$$

However, a claim

$$\forall^* x \in S^\infty \; \forall s \in S \; \exists n \;\; x(n) = s$$

fails badly for uncountable S. We observe that quantifiers $\forall^*$ and $\forall$ need not commute. On the other hand, they commute when the "$\forall$" quantifier has a countable range:

$$\forall n \, \forall^* x \;\; (\dots) \quad \Longleftrightarrow \quad \forall^* x \;\; \forall n \;\; (\dots)\,,$$

that is,

$$\forall n \, \forall^* x \;\; x \in A_n \quad \Longleftrightarrow \quad \forall^* x \;\; \forall n \;\; x \in A_n\,,$$

since all A_n are comeager if and only if $\cap_n A_n$ is comeager.

Given $x \in S^\infty$, we denote for brevity

$$x(1, 2, \dots) = \{x(n) : n = 1, 2, \dots\} \subset S\,,$$

the set of all values, an (at most) countable subset of S. By 4d5, each value is of infinite multiplicity (generically).

4d6 Exercise. The set $x(1, 2, \dots)$ is infinite (generically).

Prove it.

4d7 Exercise. For every $f : S \to S$, the set $x(1,2,\dots)$ is closed under f (generically). That is,

$$\forall f \; \forall^* x \; \forall s \; \big(s \in x(1,2,\dots) \quad \Longrightarrow \quad f(s) \in x(1,2,\dots) \big) \, .$$

Prove it.[1]

It follows for a generic $x \in (\mathbb{R}, d)^\infty$ that the set $x(1,2,\dots)$ is a countable subalgebra of $\mathbb{R}$, containing all rational numbers. An arbitrary countable set may be used instead of the rationals. For example, all algebraic numbers, or even all computable numbers. In addition, the algebra is closed under $\exp(\cdot)$, and even under all computable functions (also of several variables).

Likewise, for a Hilbert space H, for a generic $x \in (H, d)^\infty$ the set $x(1,2,\dots)$ is a countable subgroup of H; its closure is an infinite-dimensional separable subspace of H; and the subgroup is also a linear space over rational (as well as algebraic) numbers.

4d8 Exercise. Let $R \subset S \times S$ be such that $\forall s_1 \in S \; \exists s_2 \in S \; (s_1, s_2) \in R$. Then for a generic $x \in S^\infty$,

$$\forall s_1 \in x(1,2,\dots) \; \exists s_2 \in x(1,2,\dots) \; (s_1, s_2) \in R \, .$$

Prove it.

4d9 Proposition. [2] (ZF) Let A be a nonempty set and $R \subset A \times A$ satisfy $\forall a \in A \; \exists b \in A \; (a, b) \in R$. If the Baire category theorem holds for A^∞ then there exists an infinite sequence $(a_n)_n$ such that $(a_n, a_{n+1}) \in R$ for $n = 1, 2, \dots$.

Proof. The Baire theorem guarantees that the empty set is not comeager in A^∞. In combination with 4d8 this ensures existence of $x \in A^\infty$ such that

$$\forall a \in x(1,2,\dots) \; \exists b \in x(1,2,\dots) \; (a, b) \in R \, ,$$

that is,

$$\forall m \; \exists n \; \big(x(m), x(n) \big) \in R \, .$$

Denote by N_m the least n such that $\big(x(m), x(n) \big) \in R$. We take $a_n = x(k_n)$ where $(k_n)_n$ is defined recursively:

$$\begin{gathered} k_1 = 1 \, , \\ k_{n+1} = N_{k_n} \quad \text{for } n = 1, 2, \dots \, ; \end{gathered}$$

then $(a_n, a_{n+1}) = \big(x(k_n), x(N_{k_n}) \big) \in R$. □

Thus, the Baire category theorem for completely metrizable spaces is equivalent (in ZF) to the axiom of dependent choice.

[1] No, not like bosons...

[2] C.E. Blair (1977) "The Baire category theorem implies the principle of dependent choices." Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25**:10, 933–934.

Hints to exercises

4c2: (a) use 4c1; (b) use (a).

4c3: (a) similar to 4c1; (b) use (a).

4d1: (a) similar to 1d2 (but simpler).

4d5: use 4d4.

4d6: use 4d4.

4d7: use 4d4.

4d8: recall 4d7.

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