8 Fubini's theorem and Kuratowski-Ulam theorem

8a Fubini's theorem

Roughly, Fubini's theorem states that

$$\int \mathrm{d}x \int \mathrm{d}y \, f(x,y) = \iint f(x,y) \, \mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x \, f(x,y)$$

for every integrable $f : \mathbb{R}^2 \to \mathbb{R}$.

Tonelli's theorem states the same equality for every measurable $f : \mathbb{R}^2 \to [0,\infty)$; in this case the integrals belong to $[0,+\infty]$. In particular, for $f : \mathbb{R}^2 \to \{0,1\}$ we see that three measures are equal:[1]

* the two-dimensional Lebesgue measure,
* the measure $A \mapsto \int \mathrm{d}x \int \mathrm{d}y \, \mathbb{1}_A(x,y) = \int m(A^y) \, \mathrm{d}y$,
* the measure $A \mapsto \int \mathrm{d}y \int \mathrm{d}x \, \mathbb{1}_A(x,y) = \int m(A_x) \, \mathrm{d}x$;

here

$$A_x = \{ y : (x,y) \in A \}, \quad A^y = \{ x : (x,y) \in A \}.$$

These three measures are evidently equal on product sets $A = B \times C$ (be $B, C \subset \mathbb{R}$ arbitrary Lebesgue measurable sets or only intervals) and on the algebra generated by these product sets. By (the uniqueness part of) the Extension theorem[2] they are equal on the generated σ-algebra, and therefore on its completion, the two-dimensional Lebesgue σ-algebra.

[1] Measurability of the inner integrals is easy to check.

[2] "Warning: I've seen the following theorem called the Carathéodory extension theorem, the Carathéodory-Fréchet extension theorem, the Carathéodory-Hopf extension theorem, the Hopf extension theorem, the Hahn-Kolmogorov extension theorem, and many others that I can't remember! We shall simply call it Extension Theorem. However, I read in Folland's book (p. 41) that the theorem is originally due to Maurice René Fréchet (1878–1973) who proved it in 1924." Paul Loya (page 33).

All said holds on the product of two finite or σ-finite measure spaces. However, it fails for $m \times \nu$ where m is the Lebesgue measure on $\mathbb{R}$ and ν is the counting measure on $\mathbb{R}$, that is, $\nu(\{x\}) = 1$ for all $x \in \mathbb{R}$ (be it on the Borel σ-algebra, or the σ-algebra of all subsets of $\mathbb{R}$, or any intermediate σ-algebra). The measures

$$A \mapsto \int m(\mathrm{d}x) \int \nu(\mathrm{d}y)\, \mathbb{1}_A(x,y) = \int \nu(A_x)\,\mathrm{d}x = \int \mathrm{d}x \sum_y \mathbb{1}_A(x,y)\,,$$
$$A \mapsto \int \nu(\mathrm{d}y) \int m(\mathrm{d}x)\, \mathbb{1}_A(x,y) = \sum_y m(A^y)\,,$$

being equal on the algebra generated by product sets, differ on the generated σ-algebra. For example, take $A = \{(x,x) : 0 \le x \le a\}$, then $\int \nu(A_x)\,\mathrm{d}x = a$ but $\sum_y m(A^y) = 0$. By the way, the outer measure (w.r.t. the algebra generated by product sets) of A is infinite (whenever $a > 0$).

Everyone knows that Fubini's theorem is useful when calculating two-dimensional integrals. This is the "quantitative" aspect. And here is the "qualitative" aspect.

8a1 Theorem. The following three conditions on a Lebesgue measurable set $A \subset \mathbb{R}^2$ are equivalent:

(a) for almost every $x \in \mathbb{R}$ the set

$$A_x = \{y : (x,y) \in A\}$$

is a null set (in $\mathbb{R}$);

(b) A is a null set (in $\mathbb{R}^2$);

(c) for almost every $y \in \mathbb{R}$ the set

$$A^y = \{x : (x,y) \in A\}$$

is a null set (in $\mathbb{R}$).

Similarly to the category quantifiers $\forall^*$, $\exists^*$ (recall Sect. 4d) we may introduce measure quantifiers

$\forall^m x$	for almost all x
$\exists^m x$	for non-negligible set of x

and rewrite 8a1 (for the complement of A) as

$$\forall^m x\, \forall^m y\ (x,y) \in A \quad \Longleftrightarrow \quad \forall^{m\times m}(x,y)\ (x,y) \in A \quad \Longleftrightarrow \quad \forall^m y\, \forall^m x\ (x,y) \in A\,.$$

Or, equivalently (the negated claims for A itself)

$$\exists^m x\, \exists^m y\, (x,y) \in A \quad \Longleftrightarrow \quad \exists^{m\times m}(x,y)\, (x,y) \in A \quad \Longleftrightarrow \quad \exists^m y\, \exists^m x\, (x,y) \in A\,.$$

The same holds on the product of two finite or σ-finite measure spaces (but fails for $m \times \nu$).

Is this useful? Yes, it is! Here is an example from my recent work.[1,2]

> If a random compact subset K of the square $[0,1] \times [0,1]$ has almost surely uncountable first projection $\{x : \exists y \;\; (x,y) \in K\}$ then there exists a continuous function $f : [0,1] \to [0,1]$ whose graph $G_f = \{(x, f(x)) : 0 \le x \le 1\}$ meets K with positive probability.

For proving this claim I construct a *random* f that meets K with positive probability (when f and K are independent random objects) and apply 8a1 to the event $K \cap G_f \ne \emptyset$; condition (b) is violated, therefore (a) is violated; that is, the set of functions with the required property is not null, therefore not empty.

Measurability of $A \subset \mathbb{R}^2$ is crucial. The choice axiom ensures existence of a well-order "$\prec$" on $\mathbb{R}$ such that for every $y \in \mathbb{R}$ the set $\{x : x \prec y\}$ is of cardinality less than continuum. Assuming the continuum hypothesis we get a set $A = \{(x,y) : x \prec y\} \subset \mathbb{R}^2$ such that each A^y is (at most) countable, and each $\mathbb{R} \setminus A_x$ is (at most) countable. Thus, A violates 8a1(a) but satisfies 8a1(c).

8b Kuratowski-Ulam theorem

8b1 Theorem. The following three conditions on a set $A \in \mathrm{BP}(\mathbb{R}^2)$ are equivalent:

(a) for quasi all $x \in \mathbb{R}$ the set A_x is meager (in $\mathbb{R}$);
(b) A is meager (in $\mathbb{R}^2$);
(c) for quasi all $y \in \mathbb{R}$ the set A^y is meager (in $\mathbb{R}$).

That is,

$$\forall^* x\, \forall^* y \;\; (x,y) \in A \quad \Longleftrightarrow \quad \forall^*(x,y) \;\; (x,y) \in A \quad \Longleftrightarrow \quad \forall^* y\, \forall^* x \;\; (x,y) \in A\,.$$

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[1] B. Tsirelson, "Random compact set meets the graph of nonrandom continuous function", arXiv:1308.5112.

[2] Another example from my older work: a so-called spectral set is a kind of random set that contains each point with probability zero, and *therefore* is (almost surely) of zero Lebesgue measure; see B. Tsirelson, "Nonclassical stochastic flows and continuous products", page 274.

Or, equivalently,

$$\exists^* x \, \exists^* y \;\; (x,y) \in A \quad \Longleftrightarrow \quad \exists^*(x,y) \;\; (x,y) \in A \quad \Longleftrightarrow \quad \exists^* y \, \exists^* x \;\; (x,y) \in A \, .$$

The conclusion may fail on $\mathbb{R} \times (\mathbb{R}, d)$ (d being the discrete metric). Consider the closed set $A = \{(x,y) : x = y\} \subset \mathbb{R} \times (\mathbb{R}, d)$; it is nowhere dense, each A^y is meager, but no A_x is meager.

Theorem 8b1 holds on the product of two Polish spaces.

8b2 Lemma. Let X be completely metrizable and Y Polish (or just separable). If $G \subset X \times Y$ is a dense open set then $G_x \subset Y$ is a dense open set for quasi all $x \in X$.

Proof. Clearly, each G_x is open.

The projection $\{x : \exists y \;\; (x,y) \in G\}$ is a dense open set (think, why). Given a nonempty open $U \subset Y$, the set $G \cap (X \times U)$ is dense open in $X \times U$, therefore (as before) its projection is dense open in X. For quasi all x we have $\exists y \in U \;\; (x,y) \in G$, that is, $G_x \cap U \ne \emptyset$.

We take a countable base $(U_n)_n$ in Y. For quasi all x we have $\forall n \; G_x \cap U_n \ne \emptyset$, thus G_x is dense. $\square$

8b3 Exercise. Let X be completely metrizable and Y Polish.

(a) If $A \subset X \times Y$ is nowhere dense then $A_x \subset Y$ is nowhere dense for quasi all $x \in X$.

(b) If $A \subset X \times Y$ is meager then $A_x \subset Y$ is meager for quasi all $x \in X$.

(c) If $[A] = [B]$ (that is, $A \triangle B$ is meager) then $[A_x] = [B_x]$ for quasi all $x \in X$.

(d) If $A \in \mathrm{BP}(X \times Y)$ then $A_x \in \mathrm{BP}(Y)$ for quasi all $x \in X$.

Prove it.

Remark. Similarly to 8b3(d), if $A \subset \mathbb{R}^2$ is Lebesgue measurable then A_x is Lebesgue measurable for almost all $x \in \mathbb{R}$.

Proof of Theorem 8b1 (for Polish X, Y). (b)$\Longrightarrow$(a) by 8b3(b); similarly, (b)$\Longrightarrow$(c). We have to prove that (a)$\Longrightarrow$(b).

If $A \in \mathrm{BP}(X \times Y)$ is not meager then $[A] = [G]$ for some open $G \ne \emptyset$; by 8b3(c), $[A_x] = [G_x]$ for quasi all $x \in X$. The projection $G_1 = \{x : \exists y \;\; (x,y) \in G\} \subset X$ is a nonempty open set, and G_x is nonempty open for all $x \in G_1$. Thus, A_x is not meager for all x of the non-meager set G_1. $\square$

Remark. The proof of the equivalence (a)$\Longleftrightarrow$(b) uses separability of Y only. Separability of X ensures the other equivalence, (b)$\Longleftrightarrow$(c).

The Baire property of $A \subset \mathbb{R}^2$ is crucial. Once again, ZFC does not exclude existence of A such that each A^y is (at most) countable, and each $\mathbb{R} \setminus A_x$ is (at most) countable. Thus, A violates 8b1(a) but satisfies 8b1(c).

8c Some zero-one laws

Functions of the form $\limsup_n f_n \circ T_n$ appeared in Sect. 2, T being the shift on $\{0,1\}^\infty$. They are tail functions, as defined below.

Let $X_1, X_2, \dots$ be nonempty sets and $X = X_1 \times X_2 \times \dots$ their product. Consider an equivalence relation

$$x \sim y \quad \Longleftrightarrow \quad \exists n \, \forall k \;\; x(n+k) = y(n+k)$$

and the corresponding equivalence classes $[x]$ (called *germs*).

A function $f : X \to Y$ (Y being another set) is called a *tail function* if $x \sim y \implies f(x) = f(y)$.

A set $A \subset X$ is called a *tail set* if its indicator $1\!\!1_A$ is a tail function. That is,

$$x \sim y \quad \Longrightarrow \quad \bigl(x \in A \iff y \in A \bigr) .$$

Note that (f is a tail function) $\iff \forall y \in Y \; \bigl(f^{-1}(y) \text{ is a tail set} \bigr)$ $\iff \forall B \subset Y \; \bigl(f^{-1}(B) \text{ is a tail set} \bigr)$.

Every function $f : X \to \mathbb{R}$ of the form

$$f(x_1, x_2, \dots) = \limsup_n f_n(x_n, x_{n+1}, \dots)$$

(with $f_n : X_n \times X_{n+1} \times \dots \to \mathbb{R}$) is a tail function. Also, every function $f : X \to Y$ (Y being a metrizable space) of the form

$$f(x_1, x_2, \dots) = \lim_n f_n(x_n, x_{n+1}, \dots)$$

(with $f_n : X_n \times X_{n+1} \times \dots \to Y$) is a tail function.

Here is a special case of Kolmogorov's zero-one law.

8c1 Theorem. Every measurable tail set in $\{0,1\}^\infty$ is either a null set or a set of full measure.

The same holds on the product of arbitrary probability spaces.

Here is a special case of the second topological zero-one law.

8c2 Theorem. Every tail set in $\mathrm{BP}\bigl(\{0,1\}^\infty\bigr)$ is either meager or comeager.

The same holds on the product of arbitrary Polish spaces.

8c3 Exercise. Let Y be a separable metrizable space and $f : \{0,1\}^\infty \to Y$ a Borel measurable tail function. Then there exist $y_1, y_2 \in Y$ such that $f(x) = y_1$ for almost all x and $f(x) = y_2$ for quasi all x.

Deduce it from 8c1, 8c2. Can it happen that $y_1 \ne y_2$?

8c4 Exercise. (a) Using the choice axiom prove existence of a tail function $f : \{0,1\}^\infty \to \{0,1\}$ such that

$$f(1-x_1, 1-x_2, \dots) = 1 - f(x_1, x_2, \dots) \quad \text{for all } (x_n)_n \in \{0,1\}^\infty .$$

(b) Deduce from 8c1, 8c2 that such function cannot be Lebesgue measurable, and cannot have the Baire property.

Proof of Theorem 8c2 (for Polish spaces). Given n, we may treat X as $Y \times Z$ where $Y = X_1 \times \dots \times X_n$ and $Z = X_{n+1} \times X_{n+2} \times \dots$ We note that A_y does not depend on $y \in Y$ (since A is a tail set) and take $C \subset Z$ such that

$$\forall y \in Y \;\; A_y = C .$$

On the other hand, $A \in \mathrm{BP}(X)$, thus $[A] = [G]$ for some open $G \subset X$.

If $G = \emptyset$ then A is meager.

Otherwise, taking the product topology into account, we find n and a nonempty open $U \subset Y$ such that $G \supset U \times Z$; that is,

$$\forall y \in U \;\; G_y = Z .$$

By 8b3(c), $[A_y] = [G_y]$ for quasi all $y \in Y$. Thus, for quasi all $y \in U$ we have

$$[C] = [A_y] = [G_y] = [Z] .$$

We see that A_y is comeager for all y; by the Kuratowski-Ulam theorem, A is comeager. □

On the set $\{0,1\}^\infty$ we have the algebra of "elementary" (or "cylindrical", or "clopen") sets; these are the sets of the form $\{x : x[1:n] \in E\}$, $E \subset \{0,1\}^n$, $n = 1, 2, \dots$

8c5 Lemma. If $A \subset \{0,1\}^\infty$ is a measurable tail set then

$$m(A \cap B) = m(A) m(B)$$

for all elementary sets $B \subset \{0,1\}^\infty$.

Proof. Given n, we may treat $\{0,1\}^\infty$ as $Y \times Z$ where $Y = \{0,1\}^n$ and $Z = \{0,1\}^\infty$. Accordingly, $A = Y \times C$ for some $C \subset Z$; C is measurable and $m_Z(C) = m(A)$.

If $B = \tilde B \times Z$, $\tilde B \subset Y$, then $m_Y(\tilde B) = m(B)$, and $A \cap B = \tilde B \times C$, thus $m(A \cap B) = m_Y(\tilde B) m_Z(C) = m(B) m(A)$. □

Proof of Theorem 8c1. Two measures, $B \mapsto m(A \cap B)$ and $B \mapsto m(A)m(B)$, are equal (by Lemma 8c5) on the elementary algebra. By (the uniqueness part of) the Extension theorem (again!) they are equal on the generated σ-algebra (the Borel σ-algebra) and therefore on its completion, the Lebesgue σ-algebra on $\{0,1\}^\infty$.

In particular (for $B = A$), $m(A \cap A) = m(A)m(A)$, therefore $m(A)$ is either 0 or 1. $\square$

Here is another kind of zero-one laws, not related (directly) to Fubini-like theorems.

8c6 Theorem. If a measurable $A \subset \mathbb{R}$ satisfies $m\big(A \triangle (A+r)\big) = 0$ for all rational r then A is either a null set or a set of full measure.

8c7 Theorem. If a set $A \in \mathrm{BP}(\mathbb{R})$ satisfies $[A] = [A+r]$ for all rational r then A is either meager or comeager.

Proof. Recall the selector $[A] \mapsto U(A) \in [A]$ discussed in Sect. 6c; $U(A)$ is open. We have $U + r = U$ for all rational r, which evidently implies either $U = \emptyset$ or $U = \mathbb{R}$. $\square$

8c8 Exercise. Prove Theorem 8c6 by examining the measure $B \mapsto m(A \cap B)$.

Hints to exercises

8b3: 8b2$\Longrightarrow$(a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d).

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