12 Typical functions like to embed

12a A third topology on sequences

Two metrizable topologies on $[0,1]^\infty$ are mentioned in Sect. 4d. The first one is the compact product topology. The second one is the nonseparable product topology of $([0,1],d)^\infty$. Now we introduce a third one, the nonseparable topology of uniform convergence, corresponding to a complete metric

(12a1) $$\rho(x,y) = \sup_k |x(k) - y(k)| \quad \text{for } x,y \in [0,1]^\infty\,.$$

In the first topology, the set $x(1,2,\dots) = \{x(n) : n = 1,2,\dots\} \subset [0,1]$ for a typical sequence x is dense in $[0,1]$, and each point is of multiplicity 1. In the second topology, the set $x(1,2,\dots)$ typically contains all rational numbers (therefore, is dense), and each point is of infinite multiplicity. In the third topology, as we'll see soon, the set $x(1,2,\dots)$ typically is nowhere dense, and each point is of multiplicity 1.

Below, $[0,1]^\infty$ is endowed with the metric (12a1).

12a2 Lemma. $\forall t \in [0,1]\ \forall^* x \in [0,1]^\infty\ \ t \notin \operatorname{Cl}(x(1,2,\dots))$.

Proof. The function $x \mapsto \operatorname{dist}(t, x(1,2,\dots))$ on $[0,1]^\infty$ is continuous (moreover, $\operatorname{Lip}(1)$), thus, $\{x : \operatorname{dist}(t,x(1,2,\dots)) > 0\}$ is open. It is dense; indeed, $\forall x\ \forall \varepsilon\ \exists y\ \big(\rho(x,y) \le \varepsilon \wedge \operatorname{dist}(t, y(1,2,\dots)) \ge \varepsilon\big)$. □

It follows (via the Baire category theorem) that $\operatorname{Cl}(x(1,2,\dots))$ typically misses all rational numbers, and therefore is nowhere dense.

On the other hand…

12a3 Exercise. Prove that $\forall^* x \in [0,1]^\infty \;\; A \cap \operatorname{Cl}(x(1,2,\dots)) = \emptyset$
(a) whenever A is nowhere dense;
(b) whenever A is meager.

12a4 Corollary. There exists a null set $A \subset [0,1]$ such that $\forall^* x \in [0,1]^\infty$ $\operatorname{Cl}(x(1,2,\dots)) \subset A$. (Proof: just take a comeager null set.)

Given a nonempty $A \subset \{1,2,\dots\}$, we consider $x(A) = \{x(n) : n \in A\}$.

12a5 Lemma. If $A, B \subset \{1,2,\dots\}$ are disjoint then typically $\operatorname{Cl}(x(A))$ and $\operatorname{Cl}(x(B))$ are disjoint.

Proof. The function $x \mapsto \operatorname{dist}(x(A), x(B))$ is continuous (moreover, Lip(2)), and > 0 on a dense open set, since $\forall x \; \forall \varepsilon \; \exists y \; \big(\rho(x,y) \le \varepsilon \wedge \operatorname{dist}(y(A), y(B)) \ge \varepsilon \big)$; just take $y(A) \subset \{0, 2\varepsilon, 4\varepsilon, \dots\}$ and $y(B) \subset \{\varepsilon, 3\varepsilon, 5\varepsilon, \dots\}$. $\square$

Multiplicity 1 is thus ensured. Moreover, taking $A = \{n\}$ and $B = \{1,2\dots\} \setminus \{n\}$ we see that, typically, each $x(n)$ is an isolated point of $x(1,2,\dots)$.

On the other hand, $\forall x \; \exists A, B \; \big(A \cap B = \emptyset, \; \operatorname{dist}(x(A), x(B)) = 0 \big)$ (since $x(n_k) \to t$ for some $(n_k)_k$ and t).

12b Typical set of accumulation points

Consider now the space $l_\infty(\to \mathbb{R}^n)$ of all bounded sequences $x = (x(1), x(2), \dots)$ of points of $\mathbb{R}^n$, with the metric

$$\rho(x,y) = \sup_n |x(n) - y(n)| \, .$$

This is a nonseparable complete metric (moreover, Banach) space.

For each $x \in l_\infty(\to \mathbb{R}^n)$ we consider the nonempty compact set of accumulation points

$$\operatorname{Acc}(x) = \{ a : \forall \varepsilon \; \forall n \; \exists k \;\; |x(n+k) - a| \le \varepsilon \} \in \mathbf{K}(\mathbb{R}^n) \, .$$

12b1 Theorem. For quasi all $x \in l_\infty(\to \mathbb{R}^n)$ the set $K = \operatorname{Acc}(x)$ is a nowhere dense perfect null set satisfying[1]

$$\underline{\dim}_{\mathrm{M}}(K) = 0 \, , \quad \overline{\dim}_{\mathrm{M}}(K) = n \, .$$

No, we do not need to prove this from scratch. Fortunately we can use results of Sect. 10.

[1] It is also homeomorphic to the Cantor set, as we'll see in 12d.

12b2 Exercise. Let X, Y be metrizable spaces and $f : X \to Y$ be open (it means, the image of every open set is an open set) and continuous. Then the inverse image of a meager set is meager, and the inverse image of a comeager set is comeager.[1]

Prove it.

According to Remark 1f3, such f may be called genericity preserving (category preserving).

Theorem 12b1 now follows from Theorem 10c1 (and 10c2, 10c5), 12b2 and Prop. 12b3 below.

12b3 Proposition. The map

$$l_\infty(\to \mathbb{R}^n) \ni x \mapsto \operatorname{Acc}(x) \in \mathbf{K}(\mathbb{R}^n)$$

is continuous and open.

Proof. First, continuity. If $\rho(x, y) \le \varepsilon$ and $a \in \operatorname{Acc}(x)$ then $x_{n_k} \to a$ for some $(n_k)_k$, and $y_{n_{k_i}} \to b$ for some $(k_i)_i$ and b. We have $\rho(a, b) \le \varepsilon$ and $b \in \operatorname{Acc}(y)$, therefore $\operatorname{Acc}(x) \subset (\operatorname{Acc}(y))_{+\varepsilon}$. Similarly, $\operatorname{Acc}(y) \subset (\operatorname{Acc}(x))_{+\varepsilon}$. Thus, the map is continuous (and moreover, Lip(1)).

Second, openness. Let $K_1 = \operatorname{Acc}(x)$ and $d_{\mathrm{H}}(K_1, K_2) \le \varepsilon$; we have to find y close to x such that $K_2 = \operatorname{Acc}(y)$. We choose $z(1), z(2), \dots \in K_2$ such that $K_2 = \operatorname{Acc}(z)$. We take the first n_1 such that $|x(n_1) - z(1)| \le \varepsilon$ and let $y(n_1) = z(1)$. Then we take the first $n_2 > n_1$ such that $|x(n_2) - z(2)| \le \varepsilon$ and let $y(n_2) = z(2)$. And so on; $y(n_k) \in K_2$, $|y(n_k) - x(n_k)| \le \varepsilon$ and $K_2 = \operatorname{Acc}((y(n_k))_k)$. Finally, for every $n \notin \{n_1, n_2, \dots\}$ we take the first i such that $|z(i) - x(n)| \le 2\varepsilon$ and let $y(n) = z(i)$, if such i exists; otherwise $y(n) = x(n)$, but this happens only finitely many times, since $\operatorname{dist}(x_n, K_1) \to 0$. We get $\rho(x, y) \le 2\varepsilon$ and $\operatorname{Acc}(y) = K_2$. □

12b4 Exercise. Let X, Y and f be as in 12b2; assume in addition that $f(X)$ is dense in Y. Then for every $A \subset Y$, $f^{-1}(A)$ is nowhere dense if and only if A is nowhere dense.

Prove it.

12b5 Remark. Still, it can happen that $f^{-1}(A)$ is meager but A is not. An example: the projection $\mathbb{R} \times \mathbb{Q} \to \mathbb{R}$.

However, if a meager $f^{-1}(A)$ is of the form $\cup_n f^{-1}(A_n)$ with all $f^{-1}(A_n)$ nowhere dense, then A is meager.

[1] Kechris, Sect. 8K, Exer. (8.45).

12c Typical measurable function

We turn to the space $L_\infty(\to \mathbb{R}^n)$ of all equivalence classes of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}^n$, bounded (up to null sets), with the metric

$$\rho(f,g) = \operatorname{ess\,sup} |f-g| = \min\{\varepsilon : |f-g| \le \varepsilon \text{ a.e.}\} \,.$$

This is also a nonseparable complete metric (moreover, Banach) space. For each $f \in L_\infty(\to \mathbb{R}^n)$ we consider the nonempty compact set (the support)

$$\operatorname{Supp}(f) = \{a : \forall \varepsilon \;\; m(f^{-1}(\{a\}_{+\varepsilon})) > 0\} \,.$$

12c1 Exercise. $f(t) \in \operatorname{Supp}(f)$ for almost all t.
Prove it.

12c2 Proposition. The map

$$L_\infty(\to \mathbb{R}^n) \ni f \mapsto \operatorname{Supp}(f) \in \mathbf{K}(\mathbb{R}^n)$$

is continuous and open.

Proof. First, continuity. If $\rho(f,g) \le \varepsilon$ and $a \in \operatorname{Supp}(f)$ then $m(g^{-1}(a-\varepsilon-\delta, a+\varepsilon+\delta)) \ge m(f^{-1}(a-\delta, a+\delta)) > 0$ for all δ, therefore $[a-\varepsilon, a+\varepsilon] \cap \operatorname{Supp}(g) \ne \emptyset$; thus, $\operatorname{Supp}(f) \subset (\operatorname{Supp}(g))_{+\varepsilon}$. Similarly, $\operatorname{Supp}(g) \subset (\operatorname{Supp}(f))_{+\varepsilon}$. Thus, the map is continuous (and moreover, Lip(1)).

Second, openness. Let $K_1 = \operatorname{Supp}(f)$ and $d_{\mathrm{H}}(K_1, K_2) \le \varepsilon$; we have to find g close to f such that $K_2 = \operatorname{Supp}(g)$. We choose $z(1), z(2), \dots \in K_2$ such that $K_2 = \operatorname{Cl}\big(z(1,2,\dots)\big)$. We seek $g : [0.1] \to \{z(1), z(2), \dots\}$. We consider measurable sets $A_n = f^{-1}([n\varepsilon, n\varepsilon+\varepsilon))$ and for each n such that $m(A_n) > 0$ we take disjoint measurable subsets $A_{n,1}, A_{n,2}, \dots \subset A_n$ of positive measure.

For every pair n,k satisfying $|z(k) - (n+0.5)\varepsilon| \le 2\varepsilon$ we let

$$g(t) = z(k) \quad \text{for all } t \in A_{n,k} \,.$$

At least one such n exists for every k, thus all $z(k)$ belong to $\operatorname{Supp}(g)$. Also, $f(t) \in [n\varepsilon, n\varepsilon+\varepsilon)$, thus $|g(t) - f(t)| \le 3\varepsilon$.

Finally, at every other point t we let $g(t) = z(i)$ for the first i such that $|f(t) - z(i)| \le 2\varepsilon$. We get $\rho(f,g) \le 3\varepsilon$ and $\operatorname{Supp}(g) = K_2$. □

Similarly to 12b1 we get:

12c3 Theorem. For quasi all $f \in L_\infty(\to \mathbb{R}^n)$ the set $K = \operatorname{Supp}(f)$ is a nowhere dense perfect null set satisfying[1]

$$\underline{\dim}_{\mathrm{M}}(K) = 0 \,, \quad \overline{\dim}_{\mathrm{M}}(K) = n \,.$$

[1]It is also homeomorphic to the Cantor set, as we'll see in 12d.

12c4 Exercise. If $A \subset \mathbb{R}^n$ is meager then $\forall^* K \in \mathbf{K}(\mathbb{R}^n)\;\; A \cap K = \emptyset$.

Prove it.

12c5 Corollary. There exists a null set $A \subset \mathbb{R}^n$ such that $\forall^* f \in L_\infty(\to \mathbb{R}^n)\;\; \mathrm{Supp}(f) \subset A$. (Proof: just take a comeager null set.)

12c6 Exercise. If $A, B \subset [0,1]$ are disjoint measurable sets then typically $\mathrm{Supp}(f|_A)$ and $\mathrm{Supp}(f|_B)$ are disjoint.

Prove it.

12c7 Proposition. A typical $f \in L_\infty(\to \mathbb{R}^n)$ is one-to-one (that is, the equivalence class contains some one-to-one function).

Proof. We correct f on a null set getting $f(t) \in \mathrm{Supp}(f|_{[k2^{-n},(k+1)2^{-n})})$ whenever $t \in [k2^{-n}, (k+1)2^{-n})$. By 12c6 f must be one-to-one. □

Note that the dimension of $[0,1]$ is irrelevant! A typical $f \in L_\infty([0,1]^m \to \mathbb{R}^n)$ is one-to-one also when $m > n$.

Moreover, Lebesgue measure on $[0,1]$ was used only via the σ-algebra of measurable sets and the σ-ideal of null sets. All said generalizes readily to a measurable space with a given σ-ideal (under mild conditions). A measure will be more relevant in Sect. 12e.

12d Typical continuous function

A "good" function $\mathbb{R}^n \to \mathbb{R}$ behaves locally like a (nonconstant) linear function; in particular, for every Lebesgue measurable set $A \subset \mathbb{R}^n$ of positive measure,

$$f|_A \text{ is not one-to-one},$$
$$f(A) \text{ is not a null set}.$$

Let us try to imagine quite the opposite:

$$\begin{gathered} f : [0,1]^n \to \mathbb{R} \text{ is continuous,} \\ \text{and for some set } A \subset [0,1]^n \text{ of full measure}, \\ f|_A \text{ is one-to-one}, \\ f(A) \text{ is a meager set of Hausdorff dimension } 0\,. \end{gathered} \tag{12d1}$$

The latter means that for every $\varepsilon > 0$ it is possible to cover $f(A)$ with countably many balls $\{x_k\}_{+r_k}$ such that $\sum_k r_k^\varepsilon \le \varepsilon$.[1]

[1] A set of Hausdorff dimension 0 need not be meager. Moreover, it can be comeager! An example: Liouville numbers. (See Oxtoby Sect. 2 or A. Bruckner, J. Bruckner, B. Thomson "Real analysis" (second edition, 2008), Problem 10:8.3.) On the other hand, $\underline{\dim}_{\mathrm{M}}(B) < n$ implies that B is meager (and moreover, nowhere dense), just because $\underline{\dim}_{\mathrm{M}}(B) = \underline{\dim}_{\mathrm{M}}(\mathrm{Cl}(B))$.

What do you think about existence of such f?

A measurable (rather that continuous) function with similar properties[1] can be constructed using well-known tricks with digits; say (for $n = 2$)

$$f(x,y) = (0.\gamma_1\gamma_2\dots)_3 \quad \text{whenever } x = (0.\beta_1\beta_2\dots)_2\,,\ y = (0.\beta'_1\beta'_2\dots)_2\,,$$
$$\gamma_1 = 2\beta_1,\ \gamma_2 = 2\beta'_1,\ \gamma_3 = 2\beta_2,\ \gamma_4 = 2\beta'_2,\ \gamma_5 = 2\beta_3,\ \dots$$

This f is Riemann integrable (recall 5e) but has a dense set of discontinuity points. It is hard to believe that such a function can be continuous. But...

12d2 Theorem. [2] Every continuous function $[0,1]^n \to \mathbb{R}$ is the sum of two functions satisfying (12d1).

Have you any idea, why? Wait a little...

Given a metrizable space X, we consider the space $\mathrm{C}_b(X \to \mathbb{R}^n)$ of all bounded continuous functions $f : X \to \mathbb{R}^n$ with the metric

$$\rho(f,g) = \sup_{x\in X} |f(x) - g(x)|\,.$$

12d3 Proposition. Let X be a metrizable space and $Y \subset X$ a closed set. Then the map

$$\mathrm{C}_b(X \to \mathbb{R}^n) \ni f \mapsto f|_Y \in \mathrm{C}_b(Y \to \mathbb{R}^n)$$

is continuous and open.

Proof. Continuity is evident. Openness follows easily from the Tietze[-Urysohn-Brouwer-Lebesgue] extension theorem: for every $g \in \mathrm{C}_b(Y \to \mathbb{R})$ there exists $f \in \mathrm{C}_b(X \to \mathbb{R})$ such that $f|_Y = g$ and $\sup_X |f| = \sup_Y |g|$. □

It follows by 12b2 that $f|_Y$ is typical if f is typical. Thus, being interested in "very disconnected" subsets, we turn to "very disconnected" spaces.

The set Clopen(X) of all clopen (that is, open-and-closed) sets in X is an algebra of sets. If Clopen(X) $= \{\emptyset, X\}$, X is called *connected.* If Clopen(X) is a basis (of the topology), X is called *zero-dimensional.*[3] Also, X is called *perfect,* if it has no isolated points.

12d4 Lemma. A typical set of $\mathbf{K}(\mathbb{R}^n)$ is zero-dimensional.

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[1] Hausdorff dimension less than 1 (rather than 0).

[2] See also Bruckner, Bruckner, Thomson Exer. 10:7.9.

[3] If X is zero-dimensional then clearly x is *totally disconnected,* that is, contains no connected subset of more than one point. The converse holds (for compact X; and fails for some subsets of $\mathbb{R}^2$), but we do not need it.

Proof. Given $\varepsilon > 0$, consider all K such that every coordinate of every point of K belongs to $\mathbb{R} \setminus \varepsilon\mathbb{Z}$. They are a dense open set in $\mathbf{K}(\mathbb{R}^n)$, and every point has a clopen $\varepsilon\sqrt{n}$-small neighborhood. Quasi all K satisfy this condition for all $\varepsilon = 1/k$, $k = 1, 2, \dots$ □

If X is a (nonempty) perfect zero-dimensional compact (metrizable) space then clearly

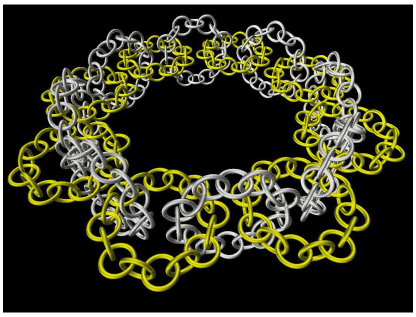
* every nonempty clopen subset of X is such space;
* for every n there exists a partition of X into n clopen sets;[1]
* for every ε there exists a finite partition of X into ε-small (that is, of diameter $\le \varepsilon$) clopen sets;
* for every ε, for every n large enough, there exists a partition of X into n ε-small clopen sets.

12d5 Lemma. All perfect zero-dimensional compact spaces are mutually homeomorphic (and therefore homeomorphic to the Cantor set).

Proof. Given such spaces X, Y, we take partitions $X = \biguplus_{k_1=1}^{n_1} X_{k_1}$, $Y = \biguplus_{k_1=1}^{n_1} Y_{k_1}$ into 1-small clopen sets. Then, partitions $X_{k_1} = \biguplus_{k_2=1}^{n_2} X_{k_1,k_2}$, $Y_{k_1} = \biguplus_{k_2=1}^{n_2} Y_{k_1,k_2}$ into 1/2-small clopen sets. And so on. Finally, we consider $G_1 = \biguplus_{k_1=1}^{n_1} X_{k_1} \times Y_{k_1} \subset X \times Y$, $G_2 = \biguplus_{k_1=1}^{n_1} \biguplus_{k_2=1}^{n_2} X_{k_1,k_2} \times Y_{k_1,k_2} \subset X \times Y$ and so on, and note that $G = \cap_n G_n$ is the graph of a homeomorphism $X \to Y$. □

12d6 Corollary. A typical set of $\mathbf{K}(\mathbb{R}^n)$ is homeomorphic to the Cantor set.

Amazingly, the Cantor set in $\mathbb{R}^n$ can be knotted! See "Antoine's necklace" in Wikipedia.[2] I wonder, is this typical?



If X is a (nonempty) compact (metrizable) space then clearly

* every nonempty closed subset of X is such space;
* for every ε there exists a finite covering of X by ε-small closed sets;
* for every ε, for every n large enough, there exists a covering of X by n ε-small closed sets. (Not necessarily different...)

[1] A partition is a covering by nonempty, pairwise disjoint sets.

[2] Image from Wikipedia.

12d7 Lemma. Every compact space is a continuous image of the Cantor set.

Proof. Let C be the Cantor set and X a compact space. We take a partition $C = \uplus_{k_1=1}^{n_1} C_{k_1}$ of C into 1-small clopen sets and a covering $X = \cup_{k_1=1}^{n_1} X_{k_1}$ of X by 1-small closed sets. Then, $C_{k_1} = \uplus_{k_2=1}^{n_2} C_{k_1,k_2}$ and $X_{k_1} = \cup_{k_2=1}^{n_2} X_{k_1,k_2}$, with 1/2-small sets. And so on. We define $G_1, G_2, \dots$ and G as before and note that G is the graph of a continuous map $C \to X$. $\square$

12d8 Proposition. The map

$$\mathrm{C}(C \to \mathbb{R}^n) \ni f \mapsto f(C) \in \mathbf{K}(\mathbb{R}^n)$$

is continuous and open.

Here C is the Cantor set, and $\mathrm{C}(C \to \mathbb{R}^n)$ is the space of all continuous maps $C \to \mathbb{R}^n$ with the metric $\rho(f,g) = \max_{x \in C} |f(x) - g(x)|$.

Proof. Continuity (and even Lip(1)) is evident; openness will be proved.

Let $K_1 = f(C)$ and $d_{\mathrm{H}}(K_1, K_2) \le \varepsilon$; we need g close to f such that $K_2 = g(C)$. We take a finite partition $C = C_1 \uplus \dots \uplus C_m$ of C into clopen sets C_k such that $\operatorname{diam}(f(C_k)) \le \varepsilon$. Sets

$$X_k = \big(f(C_k)\big)_{+\varepsilon} \cap K_2$$

are a covering of K_2 by closed sets. We take $g_k \in \mathrm{C}(C_k \to \mathbb{R}^n)$ such that $g_k(C_k) = X_k$ and combine them into $g \in \mathrm{C}(C \to \mathbb{R}^n)$, then $g(C) = K_2$ and $\rho(f,g) \le 2\varepsilon$. $\square$

Similarly to 12b1 we get:

12d9 Corollary. For quasi all $f \in \mathrm{C}(C \to \mathbb{R}^n)$ the set $K = f(C)$ is a nowhere dense null set homeomorphic to the Cantor set, satisfying $\underline{\dim}_{\mathrm{M}}(K) = 0$, $\overline{\dim}_{\mathrm{M}}(K) = n$.

12d10 Exercise. If A, B are disjoint clopen subsets of the Cantor set then typically $f(A)$ and $f(B)$ are disjoint.

Prove it.

It follows that a typical f is one-to-one. Therefore (by compactness) it is a homeomorphism between C and $f(C)$. Thus, we improve 12d9:

12d11 Theorem. For quasi all $f \in \mathrm{C}(C \to \mathbb{R}^n)$, f is a homeomorphism of C onto a nowhere dense null set $K = f(C)$ satisfying

$$\underline{\dim}_{\mathrm{M}}(K) = 0\,, \quad \overline{\dim}_{\mathrm{M}}(K) = n\,.$$

Now (at last) we are in position to attack Theorem 12d2.

12d12 Theorem. [1] There exists a set $A \subset [0,1]^n$ of full measure such that for quasi all $f \in C([0,1]^n \to \mathbb{R})$,

$$f|_A \text{ is one-to-one},$$
$$f(A) \text{ is a meager set of Hausdorff dimension } 0 .$$

A subset of $\mathbb{R}$ is zero-dimensional if and only if its complement is dense (think, why). Thus, a closed subset of $\mathbb{R}$ is zero-dimensional if and only if it is nowhere dense. By 1d4(a), the union of two zero-dimensional closed subsets of $\mathbb{R}$ is zero-dimensional.[2]

12d13 Lemma. There exist perfect zero-dimensional sets $K_n \subset [0,1]$ such that $K_1 \subset K_2 \subset \dots$ and $m(K_n) \uparrow 1$.

Proof. Monotonicity can be achieved by taking $K_1 \subset K_1 \cup K_2 \subset K_1 \cup K_2 \cup K_3 \subset \dots$ (since a finite union of perfect zero-dimensional subsets of $[0,1]$ is perfect and zero-dimensional). It remains to find, for a given ε, a perfect zero-dimensional $K \subset [0,1]$ satisfying $m(K) \ge 1 - \varepsilon$.

We take a dense sequence of pairwise disjoint closed intervals $[x_k, x_k + \delta_k] \subset [0,1]$ such that $\sum_k \delta_k \le \varepsilon$, let $K = [0,1] \setminus \cup_k (x_k, x_k + \delta_k)$ and note that K is perfect and zero-dimensional. $\square$

The same for $[0,1]^n$ follows immediately: take $K_1^n \subset K_2^n \subset \dots \subset [0,1]^n$.

12d14 Lemma. If $\underline{\dim}_{\mathrm{M}}(A) = 0$ then A is of Hausdorff dimension 0.

Proof. It is possible to cover A with $\mathcal{N}_\delta(A)$ balls of radius δ. We have $\liminf_{\delta \to 0+} \frac{\log \mathcal{N}_\delta(A)}{\log 1/\delta} = 0$. Given ε, we take δ such that $\log \mathcal{N}_\delta(A) \le \frac{1}{2} \varepsilon \log 1/\delta \le \varepsilon \log 1/\delta - \log 1/\varepsilon$, then $\delta^\varepsilon \mathcal{N}_\varepsilon(A) \le \varepsilon$. $\square$

12d15 Lemma. Sets of Hausdorff dimension 0 are a σ-ideal.

Proof. Let $A = A_1 \cup A_2 \cup \dots$, each A_k being of Hausdorff dimension 0. Given ε, for each k we cover A_k with balls $\{x_{k,i}\}_{+r_{k,i}}$ such that $\sum_i r_{k,i}^\varepsilon \le 2^{-i} \varepsilon$; then $\sum_{k,i} r_{k,i}^\varepsilon \le \varepsilon$. $\square$

Proof of Theorem 12d12. We take perfect zero-dimensional $K_1 \subset K_2 \subset \dots \subset [0,1]^n$ such that $m(K_i) \uparrow 1$ and let $A = \cup_i K_i$. By 12d5, each K_i is homeomorphic to the Cantor set. Thus, Theorem 12d11 applies to quasi all $f \in C(K_i \to \mathbb{R})$. By 12d3 (and 12b2) the same holds for quasi all $f \in$

[1] See also Bruckner, Bruckner, Thomson, Exercise 10:7.6.

[2] In more general spaces this fact holds but is harder to prove.

$C([0,1]^n \to \mathbb{R})$ restricted to K_i. That is, for each i, $f|_{K_i}$ is a homeomorphism of K_i onto a nowhere dense null set $f(K_i)$ satisfying $\underline{\dim}_{\mathrm{M}}(f(K_i)) = 0$ (and $\overline{\dim}_{\mathrm{M}}(f(K_i)) = 1$). It follows that $f|_A$ is one-to-one and $f(A)$ is meager. By 12d14, each $f(K_i)$ is of Hausdorff dimension 0. By 12d15, $f(A)$ is of Hausdorff dimension 0. □

12d16 Remark. Our choice of A ensures, in addition, that for every meager $B \subset \mathbb{R}^n$

$$\forall^* f \in C([0,1]^n \to \mathbb{R}) \;\; f(A) \cap B = \emptyset \, .$$

Thus, there exists a null set $B \subset \mathbb{R}^n$ such that

$$\forall^* f \in C([0,1]^n \to \mathbb{R}) \;\; f(A) \subset B \, .$$

(Similar to 12c4, 12c5.)

Proof of Theorem 12d2. By Theorem 12d12, quasi all $f \in C([0,1]^n \to \mathbb{R})$ satisfy (12d1). Given $g \in C([0,1]^n \to \mathbb{R})$, a map $f \mapsto g - f$ is a homeomorphism of $C([0,1]^n \to \mathbb{R})$. Thus, also $g - f$ satisfies (12d1) for quasi all f. □

12e Another topology on measurable functions

We turn to the space $L_1(\to \mathbb{R}^n)$ of all equivalence classes of Lebesgue integrable functions $f : [0,1] \to \mathbb{R}^n$ with the metric

$$\rho(f,g) = \int |f - g| \, \mathrm{d}m \, .$$

This is a Polish (in fact, Banach) space.

12e1 Lemma. $\forall x \in \mathbb{R}^n \; \forall^* f \in L_1(\to \mathbb{R}^n) \;\; m\{t : f(t) = x\} = 0$.

Proof. For every $\varepsilon > 0$ the set $\big\{ f : m\{t : f(t) = x\} < \varepsilon \big\}$ is open and dense in $L_1(\to \mathbb{R}^n)$. □

12e2 Exercise. If $A \subset \mathbb{R}^n$ is meager then $\forall^* f \in L_1(\to \mathbb{R}^n) \;\; m\big(f^{-1}(A)\big) = 0$.
Prove it.

12e3 Corollary. There exists a null set $A \subset \mathbb{R}^n$ such that for quasi all $f \in L_1(\to \mathbb{R}^n)$, $f(\cdot) \in A$ almost everywhere. (Proof: just take a comeager null set.)

Similarly to 12c we may define the support (closed rather than compact), but this time it is the whole $\mathbb{R}^n$.

12e4 Lemma. For every nonempty open $G \subset \mathbb{R}$, $\forall^* f \in L_1(\to \mathbb{R}^n)$ $m\big(f^{-1}(G)\big) > 0$.

Proof. Take continuous $\varphi : \mathbb{R}^n \to [0,\infty)$ that vanishes outside G but not everywhere. Then $f \mapsto \int \varphi(f(\cdot))\,\mathrm{d}m$ is a continuous function on $L_1(\to \mathbb{R}^n)$, positive on a dense set. □

The same holds for $f|_A$ for an arbitrary measurable $A \subset [0,1]$ of positive measure (but not for all A simultaneously, of course). Do you think it leads to infinite multiplicity? No, it does not. The result is similar to 12c7 but the proof is harder.

12e5 Proposition. A typical $f \in L_1(\to \mathbb{R}^n)$ is one-to-one (that is, the equivalence class contains some one-to-one function).

12e6 Lemma. If $A, B \subset [0,1]$ are disjoint measurable sets then for a typical $f \in L_1(\to \mathbb{R}^n)$,

$$\forall s \in A\ \forall t \in B\ \ f(s) \ne f(t)$$

for some choice of a function within the given equivalence class.

Proof. Given $\varepsilon > 0$, we introduce a set G_ε of all f such that there exist measurable $A_1 \subset A$, $B_1 \subset B$ satisfying

$$m(A \setminus A_1) < \varepsilon\,, \quad m(B \setminus B_1) < \varepsilon\,, \quad \operatorname{ess\,inf}_{s\in A_1, t\in B_1} |f(s) - f(t)| > 0\,.$$

It is sufficient to prove that a typical f belongs to all G_ε. We note that G_ε is a dense set (even for $\varepsilon = 0$) by the argument of the proof of 12a5. It remains to prove that G_ε is open (for $\varepsilon > 0$, of course).

Given $f \in G_\varepsilon$ and A_1, B_1, we take $\delta > 0$ such that $m(A \setminus A_1) \le \varepsilon - \delta$, $m(B \setminus B_1) \le \varepsilon - \delta$ and $\operatorname{ess\,inf}_{s\in A_1, t\in B_1} |f(s) - f(t)| \ge \delta$. For arbitrary $g \in L_1(\to \mathbb{R}^n)$ we have

$$m\{t : |f(t) - g(t)| \ge \delta/3\} \le \frac{3}{\delta}\|f - g\|\,.$$

If $\|f - g\| < \delta^2/3$ then the set $Z = \{t : |f(t) - g(t)| \ge \delta/3\}$ satisfies $m(Z) < \delta$. Taking $A_2 = (A\setminus A_1)\setminus Z$, $B_2 = (B\setminus B_1)\setminus Z$ we get $m(A\setminus A_2) \le m(A\setminus A_1) + \delta < \varepsilon$, $m(B \setminus B_2) < \varepsilon$, and $\operatorname{ess\,inf}_{s\in A_2, t\in B_2} |f(s) - f(t)| \ge \delta - 2\delta/3 > 0$. □

Proof of Prop. 12e5. We correct f on a null set getting $f\big([0,1/2)\big) \cap f\big([1/2,1)\big) = \emptyset$. Then we correct $f|_{[0,1/2)}$ (without increasing its image) getting $f\big([0,1/4)\big) \cap f\big([1/4,1/2)\big) = \emptyset$. And so on. □

Instead of the support, now we examine the *distribution* of f; this is a probability measure μ_f on $\mathbb{R}^n$ defined by

$$\mu_f(B) = m\big(f^{-1}(B)\big) \quad \text{for Borel sets } B \subset \mathbb{R}^n \,.$$

In general, a probability measure on $\mathbb{R}^n$ decomposes into purely atomic part (concentrated on a finite or countable set of atoms), absolutely continuous part (that has a density w.r.t. Lebesgue measure) and singular part (concentrated on an m-null set but atom-free).

By 12e5, μ_f is typically atom-free.

By 12e3, μ_f is typically singular.

Integrability of f implies $\int_{\mathbb{R}^n} |x| \, \mu_f(\mathrm{d}x) < \infty$.

The set $\mathcal{P}_1(\mathbb{R}^n)$ of all (Borel) probability measures on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} |x| \, \mu(\mathrm{d}x) < \infty$ is endowed with the so-called transportation metric

$$\rho(\mu_1, \mu_2) = \inf_{f_1, f_2 : \mu_{f_1} = \mu_1, \mu_{f_2} = \mu_2} \rho(f_1, f_2) \,.$$

Note that a sequence of purely atomic measures can converge to an absolutely continuous measure; and a sequence of absolutely continuous measures can converge to a purely atomic measure. In fact, each of the three sets of measures (purely atomic, singular, and absolutely continuous) is dense in $\mathcal{P}_1(\mathbb{R}^n)$.

12e7 Proposition. The map

$$L_1(\to \mathbb{R}^n) \ni f \mapsto \mu_f \in \mathcal{P}_1(\mathbb{R}^n)$$

is continuous and open.

Proof. Continuity (and even Lip(1)) is evident; openness will be proved.

Let $\mu_1 = \mu_{f_1}$ and $\rho(\mu_1, \mu_2) \le \varepsilon$; we need f_2 close to f_1 such that $\mu_2 = \mu_{f_2}$. We take $g_1, g_2 \in L_1(\to \mathbb{R}^n)$ such that

$$\mu_1 = \mu_{g_1} \,, \mu_2 = \mu_{g_2} \,, \quad \rho(g_1, g_2) \le 2\varepsilon \,.$$

We introduce

$$A_k = f_1^{-1}\big([k\varepsilon, k\varepsilon + \varepsilon)\big) \,, \quad B_k = g_1^{-1}\big([k\varepsilon, k\varepsilon + \varepsilon)\big)$$

for $k \in \mathbb{Z}$ and note that $m(A_k) = m(B_k)$ (since $\mu_{f_1} = \mu_{g_1}$). For each k such that $m(A_k) > 0$ we take a measure preserving map $\varphi_k : A_k \to B_k$ (try increasing φ_k such that $\forall x \;\; m\big(A_k \cap (-\infty, x]\big) = m\big(B_k \cap (-\infty, \varphi_k(x)]\big)$). We define f_2 by

$$f_2(t) = g_2(\varphi_k(t)) \quad \text{for } t \in A_k$$

and note that $\mu_{f_2} = \mu_{g_2} = \mu_2$ since for every Borel set $B \subset \mathbb{R}$,

$$m\big(f_2^{-1}(B)\big) = \sum_k m\big(f_2^{-1}(B) \cap A_k\big) = \sum_k m\{s \in A_k : g_2(\varphi_k(s)) \in B\} = \\ = \sum_k m\{t \in B_k : g_2(t) \in B\} = m\big(g_2^{-1}(B)\big) \, .$$

It remains to prove that f_2 is close to f_1. We have

$$\rho(f_1, f_2) = \int |f_1 - f_2| \, \mathrm{d}m = \sum_k \int_{A_k} |f_1 - f_2| \, \mathrm{d}m \le \\ \le \sum_k \int_{A_k} (|f_1 - k\varepsilon| + |k\varepsilon - f_2|) \, \mathrm{d}m \le \varepsilon + \sum_k \int_{A_k} (|f_2 - k\varepsilon| \, \mathrm{d}m = \\ = \varepsilon + \sum_k \int_{B_k} (|g_2 - k\varepsilon| \, \mathrm{d}m \le \varepsilon + \sum_k \int_{B_k} \big(|g_2 - g_1| + |g_1 - k\varepsilon|\big) \, \mathrm{d}m \le 2\varepsilon + \rho(g_1, g_2) \le 4\varepsilon \, .$$

$\square$

12e8 Exercise. A typical measure is atom-free.
Prove it.

12e9 Exercise. A typical measure is singular.
Prove it.

Minkowski (or “box”) dimension of a measure is defined by

$$\underline{\dim}_{\mathrm{M}} \, \mu = \liminf_{\mu(B) \to 1} \underline{\dim}_{\mathrm{M}} \, B \, , \quad \overline{\dim}_{\mathrm{M}} \, \mu = \liminf_{\mu(B) \to 1} \overline{\dim}_{\mathrm{M}} \, B$$

where B runs over all Borel sets.

It appears that[1] for quasi all $\mu \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\underline{\dim}_{\mathrm{M}} \, \mu = 0 \, , \quad \overline{\dim}_{\mathrm{M}} \, \mu = n \, .$$

By 12e7 (and 12b2, for quasi all $f \in L_1(\to \mathbb{R}^n)$,

$$\underline{\dim}_{\mathrm{M}} \, \mu_f = 0 \, , \quad \overline{\dim}_{\mathrm{M}} \, \mu_f = n \, .$$

[1] J. Myjak, R. Rudnicki (2002) “On the box dimension of typical measures”, Monatsh. Math. **136**, 1143–150.

Hints to exercises

12a3: (a) try $\operatorname{dist}(A, x(1, 2, \dots))$; (b) use (a).

12c1: 5d5 can help.

12c4: similar to 12a3.

12c6: similar to 12a5.

12d10: similar to 12a5.

12e2: recall 12a3.

12e8: no, 12e5 is of no help (I think so). Rather, prove that all μ satisfying $\forall x \;\; \mu(\{x\}) < \varepsilon$ are an open set.

12e9: use 12e3 and 12b5 if you like. Or do not.

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