5 Modes of convergence

5a Introduction

> If one has a sequence $x_1, x_2, x_3, \dots \in \mathbb{R}$ of real numbers x_n, it is unambiguous what it means for that sequence to converge to a limit $x \in \mathbb{R}$ ... If, however, one has a sequence $f_1, f_2, f_3, \dots$ of functions $f_n : X \to \mathbb{R}$ ... and a putative limit $f : X \to \mathbb{R}$ ... there can now be many different ways in which the sequence f_n may or may not converge to the limit f ... Once X becomes infinite, the functions f_n acquire an infinite number of degrees of freedom, and this allows them to approach f in any number of inequivalent ways. ... However, pointwise and uniform convergence are only two of dozens of many other modes of convergence that are of importance in analysis.
>
> Tao, Sect. 1.5 "Modes of convergence".

When speaking about functions f_n on a measure space we really mean, as usual, equivalence classes $[f_n]$. Thus we restrict ourselves to such modes of convergence that are insensitive to arbitrary change of f_n on a null set. (Do you bother whether the null set depends on n?) We replace the pointwise convergence with the convergence almost everywhere,

$$f_n \to f \text{ a.e.} \iff \mu\{x : f_n(x) \not\to f(x)\} = 0 \iff$$
$$\iff f_n \to f \text{ pointwise outside some null set,}$$

and the uniform convergence with the convergence uniform almost everywhere,

$$f_n \to f \text{ in } L_\infty \iff f_n \to f \text{ uniformly outside some null set}$$

(why "L_∞"? wait for 5f7). On the other hand, the given measure leads to new modes of convergence, such as

$$f_n \to f \text{ in } L_1 \quad \Longleftrightarrow \quad \int_X |f_n - f| \, \mathrm{d}\mu \to 0 \, .$$

5b Local convergence in measure

This is the weakest among all modes of convergence that we need. (That is, every other convergence implies the local convergence in measure.) Strangely enough, it is not mentioned by Tao (Sect. 1.5, p. 95) among 5 modes.[1]

Throughout, (X, S, μ) is a σ-finite measure space.

5b1 Definition. Let $f, f_1, f_2, \dots : X \to \mathbb{R}$ be measurable functions. We say that $f_n \to f$ *locally in measure,* if

$$\mu\{x \in A : |f_n(x) - f(x)| \ge \varepsilon\} \to 0 \quad \text{as } n \to \infty$$

for all $\varepsilon > 0$ and all $A \in S$ such that $\mu(A) < \infty$.

This definition generalizes readily to $f_n : X \to [-\infty, +\infty]$; but f must be finite a.e.

Clearly, $f_n \to f$ if and only if $f_n - f \to 0$.

5b2 Exercise. It is enough to check a single sequence of sets $A = A_1, A_2, \dots$ such that $\cup_k A_k = X$.

Prove it.[2]

If $\mu(X) < \infty$, then it is enough to check $A = X$, of course; this case is called the global convergence in measure. For $\mu(X) = \infty$ the global version is stronger than the local one; for example, on $\mathbb{R}$, $\mathbb{1}_{[n,\infty)} \to 0$ locally but not globally.

5b3 Remark. If $f_n \to f$ and $f_n \to g$ then $f = g$ a.e. (Local convergence in measure is meant in both cases.)

Proof (sketch): $\{x : |f(x) - g(x)| \ge 2\varepsilon\} \subset \{x : |f_n(x) - f(x)| \ge \varepsilon\} \cup \{x : |f_n(x) - g(x)| \ge \varepsilon\}$.

[1]Also not mentioned by Capiński & Kopp, nor by Jones. Mentioned by N. Lerner "A course on integration theory", Springer 2014 (Exercise 2.8.14 on p. 113); by F. Liese & K.-J. Miescke "Statistical decision theory", Springer 2008 (Def. A.11 on p. 619); and, prominently, on Wikipedia, "Convergence in measure".

[2]Hint: $\mu\big(A \setminus (A_1 \cup \dots \cup A_n)\big) \to 0$.

5b4 Remark (sandwich)**.** If $g_n \le f_n \le h_n$ a.e., $g_n \to f$ and $h_n \to f$, then $f_n \to f$. (Local convergence in measure is meant in all three cases.)

Proof (sketch): $\{x : |f_n(x) - f(x)| \ge \varepsilon\} \subset \{x : |g_n(x) - f(x)| \ge \varepsilon\} \cup \{x : |h_n(x) - f(x)| \ge \varepsilon\}$.

5b5 Exercise. If $f_n \to 0$ and $g_n \to 0$, then $af_n + bg_n \to 0$ for arbitrary $a, b \in \mathbb{R}$. (Local convergence in measure is meant in all three cases; no matter what is meant by $\infty - \infty$.)

Prove it.

Thus, if $f_n \to f$ and $g_n \to g$, then $af_n + bg_n \to af + bg$ (since $(af_n + bg_n) - (af + bg) = a(f_n - f) + b(g_n - g)$).

Interestingly, the local convergence in measure is insensitive not only to the choice of functions f_n in their equivalence classes, but also to the choice of a measure μ in its equivalence class defined as follows.

5b6 Definition. A measure ν on (X, S) is called *equivalent* to μ if $\nu = f \cdot \mu$ for some $f : X \to (0, \infty)$.

This is indeed an equivalence relation; symmetry: if $\nu = f \cdot \mu$ then $\mu = \frac{1}{f} \cdot \nu$; transitivity: if $\nu_1 = f \cdot \mu$ and $\nu_2 = g \cdot \nu_1$ then $\nu_2 = (fg) \cdot \mu$ (recall 4c26 and the paragraph after it).

If μ and ν are equivalent then they have the same null sets (think, why).

5b7 Lemma. If ν is equivalent to μ, then $f_n \to f$ locally in μ if and only if $f_n \to f$ locally in ν.

Proof. Let $f_n \to f$ locally in μ, $\varepsilon > 0$ and $\nu(A) < \infty$; we have to prove that $\nu(A_n) \to 0$, where $A_n = \{x \in A : |f_n(x) - f(x)| \ge \varepsilon\}$. By 5b2, WLOG we may assume that $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ are bounded on A, since countably many such sets A can cover X (think, why). Now, $\mu(A) \le \bigl(\sup_{x \in A} \frac{d\mu}{d\nu}(x)\bigr)\nu(A) < \infty$, thus $\mu(A_n) \to 0$, and $\nu(A_n) \le \bigl(\sup_{x \in A} \frac{d\nu}{d\mu}(x)\bigr)\mu(A_n) \to 0$. $\square$

5b8 Exercise. There exists a finite measure ν equivalent to μ.

Prove it.[1]

For such ν, local convergence in μ is equivalent to the global convergence in ν. The latter is easy to metrize.

5b9 Exercise. If $\mu(X) < \infty$, then the following conditions on measurable $f_n : X \to \mathbb{R}$ are equivalent:

(a) $f_n \to 0$ in measure;

[1] Hint: $X = \uplus A_k$, $\frac{d\nu}{d\mu} = \varepsilon_k$ on A_k.

(b) $\int_X \min(1, |f_n|) \, \mathrm{d}\mu \to 0$;
(c) $\int_X \frac{|f_n|}{1+|f_n|} \, \mathrm{d}\mu \to 0$.
Prove it.

We may define a metric ρ on the set[1] $L_0(X)$ of all equivalence classes of measurable functions $X \to \mathbb{R}$ by[2]

(5b10) $$\rho\big([f],[g]\big) = \int_X \min(1, |f-g|) \, \mathrm{d}\mu$$

provided that $\mu(X) < \infty$; otherwise we use a finite equivalent measure instead.

5c Convergence almost everywhere

As was noted in Sect. 4c, "almost everywhere" (a.e.) means "outside some null set". In particular, we say that $f_n \to f$ a.e., if $f_n(x) \to f(x)$ for all x outside some null set. (Sometimes this convergence is called "pointwise a.e.".)

Once again, the definition generalizes readily to $f_n : X \to [-\infty, +\infty]$; but f must be finite a.e.

Once again, $f_n \to f$ if and only if $f_n - f \to 0$. Also, if $f_n \to f$ and $g_n \to g$, then $af_n + bg_n \to af + bg$ (no matter what is meant by $\infty - \infty$).

Still, (X, S, μ) is a σ-finite measure space.

5c1 Lemma. Convergence almost everywhere implies local convergence in measure.

Proof. First, we consider monotone convergence: $f_n \downarrow 0$ a.e. In this case, given $\varepsilon > 0$ and $\mu(A) < \infty$, the sets $A_n = \{x \in A : f_n(x) \ge \varepsilon\}$ satisfy $A_n \downarrow \emptyset$, that is, $A \setminus A_n \uparrow A$ (up to a null set), which implies $\mu(A \setminus A_n) \uparrow \mu(A)$, that is, $\mu(A_n) \downarrow 0$.

Second, the general case: $f_n \to 0$ a.e. We introduce functions $g_n = \inf(f_n, f_{n+1}, \dots)$, $h_n = \sup(f_n, f_{n+1}, \dots)$; they are measurable by 3c10. Almost everywhere, $g_n \uparrow 0$ (think, why), $h_n \downarrow 0$, and $g_n \le f_n \le h_n$.[3] By the first part of this proof, $g_n \to 0$ and $h_n \to 0$ locally in measure. It remains to use 5b4. $\square$

5c2 Corollary (of 5c1 and 5b3)**.** If $f_n \to f$ locally in measure and $f_n \to g$ a.e., then $f = g$ a.e.

[1] Denoted also by $L_0(X, S, \mu)$, $L_0(\mu)$, $L^0(X)$, etc.
[2] Alternatively, use $\int_X \frac{|f-g|}{1+|f-g|} \, \mathrm{d}\mu$.
[3] Note the zigzag sandwich!

5c3 Example. Local convergence in measure does not imply convergence almost everywhere.

We take the measure space $[0,1]$ with Lebesgue measure. For every n there exists a partition of $X = [0,1]$ into n sets (just intervals, if you like) of measure $1/n$ each. We combine such partitions (for odd n) into a single infinite sequence of sets

$$\underbrace{A_1, A_2, A_3,}_{\text{partition}}\ \underbrace{A_4, A_5, A_6, A_7, A_8,}_{\text{partition}}\ \underbrace{A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15},}_{\text{partition}}\ \dots$$

such that $A_{k^2} \uplus A_{k^2+1} \uplus \dots \uplus A_{(k+1)^2-1} = X$ and $\mu(A_{k^2}) = \mu(A_{k^2+1}) = \dots = \mu(A_{(k+1)^2-1}) = \frac{1}{2k+1}$ for $k = 1, 2, \dots$ Clearly, $\mu(A_n) \to 0$, therefore the indicators $f_n = 1\!\!1_{A_n}$ converge to 0 in measure. However, $\limsup_n f_n(x) = 1$ for every x, since x belongs to infinitely many A_n.

5c4 Exercise. If X is (finite or) countable, then local convergence in measure implies convergence almost everywhere.

Prove it.

5c5 Lemma. Let $\mu(X) < \infty$, and ρ be defined by (5b10). If $\sum_n \rho(f_n, f_{n+1}) < \infty$, then the sequence $(f_n)_n$ converges a.e.

Proof. We denote $g_n = \min(1, |f_{n+1} - f_n|)$, $S_n = g_1 + \dots + g_n \uparrow S : X \to [0,\infty]$. Using the Monotone Convergence Theorem 4c11,

$$\int_X S \, \mathrm{d}\mu = \lim_n \int_X S_n \, \mathrm{d}\mu = \lim_n \Big(\int_X g_1 \, \mathrm{d}\mu + \dots + \int_X g_n \, \mathrm{d}\mu \Big) = \sum_n \int_X g_n \, \mathrm{d}\mu < \infty \, .$$

By (4c31), $S < \infty$ a.e. For almost every x we have $\sum_n \min(1, |f_{n+1}(x) - f_n(x)|) < \infty$, therefore $\sum_n |f_{n+1}(x) - f_n(x)| < \infty$ (think, why), therefore the series $\sum_n \big(f_{n+1}(x) - f_n(x)\big)$ converges, that is, the sequence $\big(f_n(x)\big)_n$ converges. $\square$

5c6 Corollary (of 5c5 and 5c1)**.** The metric space $(L_0(X), \rho)$ is complete. That is, every Cauchy sequence is converging.

It appears that convergence almost everywhere cannot be metrized (in contrast to local convergence in measure). Moreover, no functional $H : L_0[0,1] \to [0,\infty]$ satisfies

$$H(f_n) \to 0 \quad \iff \quad f_n \to 0 \text{ a.e.}$$

Proof: otherwise we take f_n such that $f_n \to 0$ in measure but not a.e.; note that $H(f_n) \not\to 0$; take a subsequence $g_i = f_{n_i}$ and $\varepsilon > 0$ such that $\forall i \;\; H(g_i) \ge \varepsilon$; note that $g_i \to 0$ in measure; take a subsequence $h_j = g_{i_j}$ such that $h_j \to 0$ a.e.; then $H(h_j) \to 0$ but $H(h_j) \ge \varepsilon$, — a contradiction.

5c7 Exercise. (a) Let $\mu(X) < \infty$ and $\forall n \;\; |f_n| \le 1$ a.e. If $f_n \to f$ locally in measure, then $\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$. Prove it.

(b) Both assumptions (about $\mu(X)$ and $|f_n|$) are essential; give two counterexamples.

5c8 Exercise. (a) A sequence of real numbers converges to 0 if and only if every subsequence contains a (sub)subsequence that converges to 0;

(b) a sequence of measurable functions $X \to [-\infty, +\infty]$ converges to 0 locally in measure if and only if every subsequence contains a (sub)subsequence that converges to 0 almost everywhere.
Prove it.

5c9 Exercise. If $f_n \to f$ locally in measure, then $\varphi \circ f_n \to \varphi \circ f$ locally in measure for every continuous $\varphi : [-\infty, +\infty] \to [-\infty, +\infty]$.

Prove it.[1]

5d Dominated convergence; integrals depending on a parameter

5d1 Theorem (Dominated Convergence Theorem)**.** Let $f, f_1, f_2, \dots : X \to \mathbb{R}$ and $g : X \to [0, \infty)$ be measurable functions such that

$$\begin{gathered} f_n \to f \text{ a.e.}\,; \\ \forall n \;\; |f_n| \le g \text{ a.e.}\,; \\ g \text{ is integrable}\,. \end{gathered}$$

Then f is integrable, and

$$\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu \,.$$

Proof. We did not stipulate σ-finiteness of μ, but this does not matter; WLOG, $g : X \to (0, \infty)$, since all f_n and f vanish outside $\{x : g(x) \ne 0\}$; now μ must be σ-finite, since it is equivalent to the finite measure $\nu = g \cdot \mu$. By (4c28),[2]

$$\int_X f_n \, \mathrm{d}\mu = \int_X \frac{f_n}{g} \, \mathrm{d}\nu \,, \qquad \int_X f \, \mathrm{d}\mu = \int_X \frac{f}{g} \, \mathrm{d}\nu \,.$$

It remains to use 5c7 (and 5c1). $\square$

[1] Hint: use 5c8.
[2] Generalized to integrable (signed) functions.

5d2 Remark. The a.e. convergence ($f_n \to f$) may be replaced with the local convergence in measure. (Use 5c8.)

The Dominated Convergence Theorem 5d1 is useful when dealing with integrals depending on a parameter.[1] Consider a function $f : \mathbb{R} \times X \to \mathbb{R}$ such that its first section $f(t,\cdot) : x \mapsto f(t,x)$ is integrable (for every $t \in \mathbb{R}$), and its second section $f(\cdot,x) : t \mapsto f(t,x)$ is continuous (for every $x \in X$). We introduce $F(t) = \int_X f(t,\cdot)\,\mathrm{d}\mu$ and wonder, whether F is continuous, or not.

The function $g : x \mapsto \sup_{t\in\mathbb{R}} |f(t,x)|$ is measurable (since the supremum may be taken over rational t); and if g is integrable, then F is continuous (since $t_n \to t$ implies $F(t_n) \to F(t)$ by 5d1).

(5d3) If $t \mapsto f(t,x)$ is continuous and $\displaystyle\int_X \sup_t |f(t,\cdot)|\,\mathrm{d}\mu < \infty$,

then $\displaystyle t \mapsto \int_X f(t,\cdot)\,\mathrm{d}\mu$ is continuous.

(Warning: $\int \sup(\dots)$ is generally larger than $\sup \int(\dots)$.)

Moreover, it is enough if $t \mapsto f(t,x)$ is continuous at t for all x outside some null set that may depend on t. In particular,

(5d4) if $f : \mathbb{R} \to \mathbb{R}$ is integrable, then $\displaystyle t \mapsto \int_0^t f\,\mathrm{d}m$ is continuous.

(Consider $\tilde{f}(t,x) = f(x)1\!\!1_{[0,t]}(x)$.) Similarly, for integrable (and maybe, nowhere continuous) $f : \mathbb{R}^d \to \mathbb{R}$ the function $r \mapsto \int_{|\cdot|<r} f\,\mathrm{d}m$ is continuous. The same holds for $r \mapsto \int_{|\cdot|<r} f(r,\cdot)\,\mathrm{d}m$ when $r \mapsto f(r,x)$ is continuous for all x, provided that the supremum is integrable.

When (X,S,μ) is a probability space, we get a probabilistic statement:

(5d5) If a random function is continuous almost surely (at each point separately) and the supremum of its absolute value has finite expectation, then its expectation is a continuous function.

5d6 Example. The integrable majorant is essential.

Consider a random function equal to $2^n f(2^n t)$ with probability 2^{-n}, $n = 1,2,\dots$; here

$$f(t) = \begin{cases} 2t-1 & \text{for } t \in [\frac12, 1], \\ 2-t & \text{for } t \in [1,2], \\ 0 & \text{for } t \in (-\infty, \frac12] \cup [2,\infty). \end{cases}$$

[1]See Jones, Sect. 6G.

Its expectation is

$$\sum_{n=1}^{\infty} f(2^n t) = \begin{cases} 0 & \text{for } t \in (-\infty, 0] \cup [1, \infty), \\ 1 & \text{for } t \in (0, \frac{1}{2}], \\ 2(1-t) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

since for $t = 2^{-n}\theta$, $\theta \in [\frac{1}{2}, 1]$ we have $f(2^n t) = 2\theta - 1$ and $f(2^{n+1}t) = 2 - 2\theta$; other summands vanish. The expectation of the supremum is $\sum_{n=1}^{\infty} 2^n 2^{-n} = \infty$.

Now we wonder, whether $F : t \mapsto \int_X f(t, \cdot) \, \mathrm{d}\mu$ is continuously differentiable. We assume that, for almost every x, $f(\cdot, x) \in C^1(\mathbb{R})$; and in addition, that the function $g : x \mapsto \sup_{t \in \mathbb{R}} |\frac{\partial}{\partial t} f(t, x)|$ is integrable. Then, by (5d3), $t \mapsto \int_X \frac{\partial}{\partial t} f(t, \cdot) \, \mathrm{d}\mu$ is continuous. But is it F'? We have

$$\frac{f(t+h, \cdot) - f(t, \cdot)}{h} \to \frac{\partial}{\partial t} f(t, \cdot) \text{ a.e.,} \qquad \Big| \frac{f(t+h, \cdot) - f(t, \cdot)}{h} \Big| \le g(\cdot) \, ;$$

by 5d1, $\frac{F(t+h)-F(t)}{h} = \int_X \frac{f(t+h,\cdot)-f(t,\cdot)}{h} \, \mathrm{d}\mu \to \int_X \frac{\partial}{\partial t} f(t, \cdot) \, \mathrm{d}\mu$ as $h \to 0$. Nice; but is $f(t, \cdot)$ integrable? For now we only know that $f(t+h, \cdot) - f(t, \cdot)$ is integrable for all t and h.

(5d7) If $t \mapsto f(t, x) \in C^1(\mathbb{R})$, and $\displaystyle\int_X \sup_{t \in \mathbb{R}} \Big| \frac{\partial}{\partial t} f(t, \cdot) \Big| \, \mathrm{d}\mu < \infty$, and $f(t, \cdot)$ is integrable for some (therefore, every) t, then

$$t \mapsto \int_X f(t, \cdot) \, \mathrm{d}\mu \in C^1(\mathbb{R}) \, , \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_X f(t, \cdot) \, \mathrm{d}\mu = \int_X \frac{\partial}{\partial t} f(t, \cdot) \, \mathrm{d}\mu \, .$$

The corresponding probabilistic statement:

(5d8) If a random function belongs to $C^1(\mathbb{R})$ almost surely,
and the supremum of absolute value of its derivative has finite expectation,
and the expectation of the function is well-defined at some (therefore, every) point,
then the derivative of the expectation is equal to the expectation of the derivative,
and is continuous.

5e One-sided bounds

First, the monotone case.

In the Monotone Convergence Theorem 4c11, functions f_n are bounded from below by 0; equally well they may be bounded from below by an integrable function.

5e1 Corollary (of 4c11)**.** Let $g, f, f_1, f_2, \dots : X \to [-\infty, +\infty]$ be measurable functions such that $f_n \uparrow f$ a.e., $f_1 \ge g$ and g is integrable. Then

$$\int_X f_n \, \mathrm{d}\mu \uparrow \int_X f \, \mathrm{d}\mu \in (-\infty, +\infty] \, .$$

For the proof, just apply 4c11 to $(f_n - g) \uparrow (f - g)$ and cancel $\int g$.

The integrable minorant g for f_1 exists if and only if $\int_X f_1^- \, \mathrm{d}\mu < \infty$ (think, why).

Clearly, 5e1 still holds if $\exists n \;\; \int_X f_n^- \, \mathrm{d}\mu < \infty$. Otherwise, if $\forall n \;\; \int_X f_n^- \, \mathrm{d}\mu = \infty$, 5e1 fails for two reasons. First, it may happen that $\int_X f_n \, \mathrm{d}\mu$ is $\infty - \infty$ for all n, and the conclusion is a nonsense. Second, even if $\int_X f_n \, \mathrm{d}\mu = -\infty$ for all n, it does not imply $\int_X f \, \mathrm{d}\mu = -\infty$. For a counterexample take a nonintegrable $h : X \to [0, \infty)$, an integrable f, and consider $f_n = f - \frac{1}{n} h$.

Now, the non-monotone case.

In the Dominated Convergence Theorem 5d1, functions f_n are bounded by integrable function(s) from both sides: $-g \le f_n \le g$. In Example 5d6, taking $t_n \to 0$, we get $f_n \to 0$ a.e., but $\int_X f_n \, \mathrm{d}\mu \to 1$; these f_n are bounded from below (just by 0), and surely not bounded from above (by an integrable function) according to 5d1. Also, $(-f_n) \to 0$ a.e., but $\int_X (-f_n) \, \mathrm{d}\mu \to -1$; and $(-f_n) \le 0$. Does it mean that $\lim_n \int f_n$ can deviate from $\int \lim_n f_n$ only to the unbounded direction? Can we prove that $\lim_n \int_X f_n \, \mathrm{d}\mu \le \int_X f \, \mathrm{d}\mu$ whenever $f_n \to f$ a.e. and $\forall n \;\; f_n \le g$, g integrable, or equivalently, $\int_X (\sup_n f_n)^+ \, \mathrm{d}\mu < \infty$? No, we cannot, since $\lim_n \int_X f_n \, \mathrm{d}\mu$ need not exist (try $f_2 = f_4 = \dots = f$). But we can prove that $\limsup_n \int_X f_n \, \mathrm{d}\mu \le \int_X f \, \mathrm{d}\mu = \int_X \big(\lim_n f_n \big) \, \mathrm{d}\mu$. Moreover, we do not really need existence of $\lim_n f_n$; we can prove that $\limsup_n \int_X f_n \, \mathrm{d}\mu \le \int_X \big(\limsup_n f_n \big) \, \mathrm{d}\mu$ whenever $\big(\sup_n f_n \big)^+$ is integrable. Or we may change the signs of f_n.

5e2 Proposition (Fatou's Lemma)**.**

$$\int_X \big(\liminf_n f_n \big) \, \mathrm{d}\mu \le \liminf_n \int_X f_n \, \mathrm{d}\mu$$

whenever measurable functions $f_n : X \to [-\infty, +\infty]$ are such that $(\inf_n f_n)^-$ is integrable.

(If $f_n : X \to [0, \infty]$, then $(\inf_n f_n)^- = 0$ is integrable, of course.)

5e3 Remark. Let (X, S, μ) be a probability space, $A_1, A_2, \dots \in S$, $A_n \downarrow \emptyset$, and $\mu(A_n) = \frac{1}{n}$. Consider $f_n = n 1\!\!1_{A_n}$. Clearly, $f_n \to 0$ a.s., but $\forall n \;\; \int_X f_n \, \mathrm{d}\mu = 1$. The distribution of (the random variable) f_n (recall page 30) consists of two atoms; one atom of mass $1/n$ at n, the other atom of mass $1 - \frac{1}{n}$ at 0. When n tends to infinity, the first atom "escapes to infinity". That is, its contribution to the expectation (the integral) escapes to ∞; its mass (probability) does not escape, it returns to (the atom at) 0.

The integrable lower bound prevents escape to $-\infty$, but does not prevent escape to $+\infty$. Intuitively, Fatou's Lemma 5e2 says that a part of the integral can escape to infinity (in the allowed direction), but cannot come from infinity.

5e4 Remark. Nothing like that holds for $\limsup_n f_n$ (unless we turn to integrable *upper* bound). Indeed, according to 5c3 it can happen that $f_n \ge 0$, $\int_X f_n \, \mathrm{d}\mu \to 0$, but $\limsup_n f_n = 1$ a.e.

***Proof** of Prop. 5e2.* We introduce[1] functions $g_n = \inf(f_n, f_{n+1}, \dots)$, note that $g_n \uparrow \liminf_n f_n$, and apply Monotone Convergence Theorem 4c11 to functions $g_n - g$, where $g = -(\inf_n f_n)^-$ is an integrable lower bound of all f_n and therefore of all g_n, too; we get (canceling $\int g$)

$$\int_X g_n \, \mathrm{d}\mu \uparrow \int_X (\liminf_n f_n) \, \mathrm{d}\mu \, .$$

Thus,

$$\int_X \bigl(\liminf_n f_n\bigr) \, \mathrm{d}\mu = \lim_n \int_X g_n \, \mathrm{d}\mu = \liminf_n \int_X g_n \, \mathrm{d}\mu \le \liminf_n \int_X f_n \, \mathrm{d}\mu \, ,$$

since $g_n \le f_n$. $\square$

5e5 Exercise.

$$\int_X f \, \mathrm{d}\mu \le \liminf_n \int_X f_n \, \mathrm{d}\mu$$

whenever measurable functions $f, f_n : X \to [-\infty, +\infty]$ are such that $f_n \to f$ locally in measure and $(\inf_n f_n)^-$ is integrable.

Prove it.[2]

5e6 Corollary. (a) The set of all $f \in L_0(X)$ such that $f \ge 0$ and $\int_X f \, \mathrm{d}\mu \le 1$ is closed in $L_0(X)$ (w.r.t. the local convergence in measure);

(b) the same holds for f such that $\int |f| \, \mathrm{d}\mu \le 1$, and more generally, for f such that $\int \varphi \circ f \, \mathrm{d}\mu \le 1$ for a given continuous $\varphi : [-\infty, +\infty] \to [0, \infty]$ (recall 5c9).[3]

In contrast, the set of all $f \in L_0(X)$ such that $f \ge 0$ and $\int_X f \, \mathrm{d}\mu \ge 1$ is not closed in $L_0(X)$.

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[1]Similarly to the proof of 5c1.

[2]Hint: 5c8.

[3]Still more generally: for all lower semicontinuous φ (it means, $\lim_{s \to t} \varphi(s) \ge \varphi(t)$).

5f Spaces L_p

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a *convex* function such that $\varphi(0) = 0$, $\varphi(t) > 0$ for all $t \ne 0$, and $\varphi(-t) = \varphi(t)$. For example, $\varphi(t) = |t|^p$ for a given $p \in [1, \infty)$.

We introduce the set

$$B_\varphi = \Big\{ f \in L_0(X) : \int_X \varphi \circ f \, \mathrm{d}\mu \le 1 \Big\} .$$

By 5e6(b), B_φ is closed in $L_0(X)$. Clearly, B_φ is symmetric: $(-f) \in B_\varphi \iff f \in B_\varphi$. Also, B_φ is convex:

$$\theta f + (1-\theta) g \in B_\varphi \quad \text{for all } f, g \in B_\varphi \text{ and } \theta \in [0,1] ,$$

since $\int_X \varphi\big(\theta f + (1-\theta)g\big) \, \mathrm{d}\mu \le \int_X \big(\theta\varphi(f) + (1-\theta)\varphi(g)\big) \, \mathrm{d}\mu = \theta \int_X \varphi(f) \, \mathrm{d}\mu + (1-\theta) \int_X \varphi(g) \, \mathrm{d}\mu \le \theta + (1-\theta) = 1$.

In particular, $rB_\varphi \subset B_\varphi$ for $r \in [0,1]$.

We introduce the functional $\| \cdot \|_\varphi : L_0(X) \to [0, \infty]$ by

$$\|f\|_\varphi = \inf\{ r > 0 : f \in rB_\varphi \} = \inf\{ r > 0 : \tfrac{1}{r} f \in B_\varphi \} ;$$

the infimum is reached (since B_φ is closed) unless the set is empty (in which case $\|f\|_\varphi = \inf \emptyset = +\infty$, of course). Convexity of B_φ implies the triangle inequality:

$$\|f + g\|_\varphi \le \|f\|_\varphi + \|g\|_\varphi ,$$

since, assuming $\|f\|_\varphi + \|g\|_\varphi < \infty$ (otherwise nothing to prove) we take $\theta \in [0,1]$ such that $\|f\| = \theta(\|f\| + \|g\|)$, $\|g\| = (1-\theta)(\|f\| + \|g\|)$ and get $f \in \theta(\|f\|+\|g\|)B_\varphi$, $g \in (1-\theta)(\|f\|+\|g\|)B_\varphi$, whence $f+g \in (\|f\|+\|g\|)B_\varphi$.

Clearly, $\|f\|_\varphi = 0$ if and only if $f = 0$ a.e.[1]

Thus, the set

$$L_\varphi(X) = \{ f \in L_0(X) : \|f\|_\varphi < \infty \}$$

is a vector space; $nB_\varphi \uparrow L_\varphi(X)$ as $n \to \infty$. Being endowed with the norm $\| \cdot \|_\varphi$ it is a normed space, and moreover, a Banach space by 5f3 below.

In particular, when $\varphi(t) = |t|^p$ for a given $p \in [1, \infty)$, we get the space[2] $L_p(X)$, with the norm $\| \cdot \|_p$,

$$\|f\|_p = \Big(\int_X |f|^p \, \mathrm{d}\mu \Big)^{1/p} .$$

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[1] If in trouble with the proof, read Lemma 5f1 below.

[2] Denoted also by $L_p(X, S, \mu)$, $L_p(\mu)$, $L^p(X)$, etc.

5f1 Lemma. If $f_n \to 0$ in $L_\varphi(X)$ (that is, $f_n \in L_\varphi(X)$ and $\|f_n\|_\varphi \to 0$), then $f_n \to 0$ in measure (globally, and therefore locally).

Proof. By convexity, $t \mapsto \varphi(t)/t$ is increasing on $(0,\infty)$, therefore $\varphi(t) \uparrow \infty$ as $t \to \infty$. We have $f_n \in r_n B_\varphi$, $r_n \to 0$. For every $\varepsilon > 0$,

$$\mu\{x : |f_n(x)| \ge \varepsilon\} = \mu\Big\{x : \frac{|f_n(x)|}{r_n} \ge \frac{\varepsilon}{r_n}\Big\} = \mu\Big\{x : \varphi\Big(\frac{|f_n(x)|}{r_n}\Big) \ge \varphi\Big(\frac{\varepsilon}{r_n}\Big)\Big\} \le \\ \le \frac{1}{\varphi(\varepsilon/r_n)} \int_X \varphi\Big(\frac{1}{r_n} f_n\Big)\,\mathrm{d}\mu \le \frac{1}{\varphi(\varepsilon/r_n)} \to 0\,.$$

□

Let μ be σ-finite, ν a finite measure equivalent to μ, and ρ the corresponding metric on $L_0(X)$ (introduced in (5b10)).

5f2 Exercise. (a) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall f \in L_0(X) \ \big(\|f\|_\varphi < \delta \quad\Longrightarrow\quad \rho(f,0) < \varepsilon \big)\,.$$

(b) Every Cauchy sequence in $L_\varphi(X)$ is a Cauchy sequence in $\big(L_0(X),\rho\big)$. Prove it.[1]

5f3 Proposition. The normed space $L_\varphi(X)$ is complete (that is, every Cauchy sequence is converging).

Proof. Let $f_1, f_2, \dots \in L_\varphi(X)$ be a Cauchy sequence, that is, $\sup_{k,l\ge n} \|f_k - f_l\|_\varphi = \varepsilon_n \downarrow 0$. We did not stipulate σ-finiteness of μ, but this does not matter (similarly to the proof of 5d1); WLOG, μ is σ-finite, since all f_n vanish outside the countable union $\cup_{m,n}\{x : |f_n| > 1/m\}$ of sets of finite measure. By 5f2(b), f_n are also a Cauchy sequence in $\big(L_0(X),\rho\big)$. By 5c6 there exists $f \in L_0(X)$ such that $f_n \to f$ in $L_0(X)$. We note that $f_{n+k} - f_n \in \varepsilon_n B_\varphi$, and $\varepsilon_n B_\varphi$ is closed in $L_0(X)$ (think, why), therefore $f - f_n \in \varepsilon_n B_\varphi$, that is, $\|f_n - f\|_\varphi \le \varepsilon_n \to 0$. □

Thus, all $L_p(X)$ for $p \in [1,\infty)$ are Banach spaces. But $L_2(X)$ is, moreover, a Hilbert space. If $f, g \in L_2(X)$, then $f+g \in L_2(X)$, thus, $f^2, g^2, (f+g)^2$ and $2fg = (f+g)^2 - f^2 - g^2$ are integrable. We introduce the *inner product*

$$\langle f, g\rangle = \int_X fg\,\mathrm{d}\mu \quad \text{for } f,g \in L_2(X)\,;$$

[1] Hint: (a) use 5f1; (b) use (a).

it is bilinear, symmetric, and $\langle f, f\rangle = \|f\|_2^2$, which shows that $L_2(X)$ is a Hilbert space. Every finite-dimensional subspace of $L_2(X)$ is a Euclidean space, therefore, linearly isometric to $\mathbb{R}^n$ with the Euclidean norm $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$. In particular, every 2-dimensional subspace is a Euclidean plane. Applying this fact to the subspace spanned by f, g we get $\langle f, g\rangle = \|f\|_2\|g\|_2 \cos\theta$ for some θ (the angle between the vectors f, g), and therefore

$$-\|f\|_2\|g\|_2 \le \langle f, g\rangle \le \|f\|_2\|g\|_2$$

(Schwarz inequality).

Back to $L_p(X)$.

5f4 Lemma. If $\mu(X) < \infty$ and $1 \le p \le q < \infty$, then $L_p(X) \supset L_q(X)$ and

$$\sup_{\|f\|_q=1} \|f\|_p < \infty\,.$$

Proof. [1] If $\|f\|_q = 1$, then $\int_X |f|^q \, d\mu = 1$; we note that $|f|^p \le \max(1, |f|^q) \le 1 + |f|^q$ (think, why), and get $\int_X |f|^p \, d\mu \le \mu(X) + \int_X |f|^q \, d\mu = \mu(X) + 1$, that is, $\|f\|_p \le \bigl(\mu(X) + 1\bigr)^{1/p}$. $\square$

5f5 Remark. (a) On $(0, 1)$ (with Lebesgue measure), the function $t \mapsto t^{-\alpha}$ belongs to L_p if and only if $p < 1/\alpha$ (check it), which shows that the inclusion in 5f4 is generally strict: $L_p(X) \supsetneqq L_q(X)$.

(b) On $(1, \infty)$ (with Lebesgue measure), the function $t \mapsto t^{-\alpha}$ belongs to L_p if and only if $p > 1/\alpha$ (check it), which shows that the inclusion in 5f4 may fail when $\mu(X) = \infty$.

5f6 Lemma. If X is countable and $\inf_{x\in X} \mu(\{x\}) > 0$, then $L_p(X) \subset L_q(X)$ for $1 \le p \le q < \infty$, and

$$\sup_{\|f\|_p=1} \|f\|_q < \infty\,.$$

Proof. If $\|f\|_p = 1$, then $\sum_{x\in X} |f(x)|^p \mu(\{x\}) = \int_X |f|^p \, d\mu = 1$. Introducing $\varepsilon = \inf_{x\in X} \mu(\{x\}) > 0$ we note that $|f(x)| \le \bigl(\frac{1}{\varepsilon}\bigr)^{1/p}$, therefore $|f|^q \le \bigl(\frac{1}{\varepsilon}\bigr)^{(q-p)/p} |f|^p$, and get $\int_X |f|^q \, d\mu \le \bigl(\frac{1}{\varepsilon}\bigr)^{(q-p)/p} \int_X |f|^p \, d\mu = \bigl(\frac{1}{\varepsilon}\bigr)^{(q-p)/p}$, that is, $\|f\|_q \le \bigl(\frac{1}{\varepsilon}\bigr)^{\frac{1}{p}-\frac{1}{q}}$. $\square$

5f7 Exercise. (a) $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$, where

$$\|f\|_\infty = \operatorname{ess\,sup}_{x\in X} |f(x)| = \inf\{t \in (0, \infty) : |f(\cdot)| \le t \text{ a.e.}\}$$

[1] See Capiński & Kopp, Th. 5.25. In fact, the supremum is reached on constant f and is therefore equal to $(\mu(X))^{\frac{1}{p}-\frac{1}{q}}$.

(the infimum is reached, unless the set is empty, in which case $\|f\|_\infty = \inf \emptyset = +\infty$, of course).

(b) If $\mu(X) < \infty$ then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

(c) The set
$$L_\infty(X) = \{f \in L_0(X) : \|f\|_\infty < \infty\},$$
endowed with $\|\cdot\|_\infty$, is a Banach space.

Prove it.[1] Try to extend 5f4 and 5f6 to $q = \infty$.

5f8 Remark. All these modes of convergence lead to the same equivalence classes, in the following sense: if $f_1 = f_2 = \dots$, then $f_n \to f$ holds if and only if $f_1 \sim f$.

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[1] Hint: (a) $\|f\|_p \ge c\big(\mu\{x : |f(x)| \ge c\}\big)^{1/p}$; (b) use (a); (c) 5c6 aside, use uniform convergence outside a null set.