Given a sequence $(x_1, \dots, x_n)$ of digits $x_k \in \{0, 1, \dots, 9\}$, we consider sets

$$E_d = \{k : x_k = 0, x_{k+d} = 1, x_{k+2d} = 2, x_{k+3d} = 3, x_{k+4d} = 4\} ;$$

if $k + id > n$ it is treated modulo n (that is, $n + 1 = 1$ and so on).

Let $n = 10\,000$, and $x_1, \dots, x_n$ be chosen at random (probability 10^{-n} for each sequence).

Clearly, $\mathbb{E}\,|E_d| = 0.1$ for each d; here $|E_d|$ is the number of elements in E_d.

Question 1. Prove that

$$\mathbb{P}\big(50 < |E_1| + \dots + |E_{1\,000}| < 150\big) > 0.95\,.$$

Consider events

$$\begin{aligned} A &= \{x_1 = 0\}\,, \\ B &= \{|E_1| = 0, \dots, |E_{11}| = 0, |E_{12}| = 1\}\,, \\ C &= \big\{E_1 = \emptyset, \dots, E_{11} = \emptyset, E_{12} = \{1\}\big\}\,. \end{aligned}$$

Question 2. Prove that

$$\mathbb{P}\big(C \big| A \cap B\big) = \frac{10}{n} \frac{\mathbb{P}\big(B\big)}{\mathbb{P}\big(B \big| A\big)}\,.$$

Question 3. Prove that

$$|\mathbb{P}\big(B \big| A\big) - \mathbb{P}\big(B\big)| \le 0.015\,.$$